Music and Math: Combining Forms and Formulas

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Overview

Math with its abstract formulas and unique vocabulary, seems obscure at best and most often irrelevant to students. Before students enter high school they often wonder how school and the subjects they are learning relate to the real world. During their high school years students remain disconnected from math in their daily lives. Academically, math is a subject with the lowest achievement indicators particularly when measured by standardized tests. Finding ways to make math relevant and engage students in ways that will increase math achievement is an educational goal both in the classroom and in the larger educational community.

By contrast, music is a constant in students' lives and its many forms are listened to by young people of all ages. Students are listening to music, downloading music, sharing music and watching music performances in many formats and on multiple platforms. Music is exciting, relevant and a medium that brings joy to listeners. Using music, with its natural relationship to math, will provide an engaging, fun way to learn and remember math while applying music concepts. This unit includes lessons combining music and math. It explains music form and compares it to mathematical formulas. The lessons are designed for high school mathematics students enrolled in Algebra 1 or Algebra 2 who are exploring parent functions and families of functions. Rather than limiting learning to formulas the unit connects students to math through the music and in the process, emphasizes the importance of education across two disciplines: music and mathematics.

Rationale

The seminar, “That’s My Song! Musical Genre as Social Contract”, examined musical genre and the role music plays in both our lives and society. This seminar examined how various musical genres served as “social contracts” among audiences throughout our country’s nation building process. We viewed America’s melting pot ideal by looking at how communities of listeners have asserted their convictions about social identity through music. In each class as music was discussed we learned to listen to its structure with an ear for timbre, form and the many rhythms and messages conveyed by the music. The idea for combining the concept of musical form with mathematical formulas began within our classroom discussion of the 12-bar jazz/blues form. I investigated how music forms and mathematical formulas could be used as structures and be combined and used with musical clips to aid understanding and increase retention of mathematical function families. Rather than limiting learning to formulas the unit connects students to math through the music. In the process, the lessons emphasize the importance of education across two disciplines: music and mathematics.

In this seminar, when we discussed the 12-bar jazz/blues music form and listened to music to specifically identify the form carried out in the music we became music students immersed in a new concept. For the musician, music form is just the ordering of ideas, concepts or values. Blues has a form, Sonata has a form, minuet has a form, Rondo has a form, double exposition concerto has a form, R & B has a form, rap and other pop styles have a form. The form of symphonies and operas for large-scale works are very different but can be related to math as a teacher looks at teaching formulas and the structures the forms dictate for different function families. Talking about the order in music and relating it to math means identifying variables and specific content values. For math, if you have a specific formula in mind, breaking it down into the different parts and then relating the parts and the formula to connect with form in music will help students make connections and relate a concrete musical idea with the abstract math concept. The class maybe able to feel the cadence of the song and the math and find ways to internalize the math using music. Connecting formulas to a song will involve understanding that the form for the song in the big parts is also the form for an equation. Talking about the order in music and relating it to math means identifying variables and specific content values and relating them to music.

In today's world, the subjects’ music and math don't have much to do with one another in our common knowledge and conceptions of the subjects. This conception, however is a historical one. Some of the greatest thinkers of human history, who were called philosophers in their time, dedicated countless hours of their time and energy to interrogating the principles of what they believed to be the equivalent sciences of math and music. (Pickover, 2009) These philosophers go back to as early as 1500 BCE and civilizations as old as Ancient Greece and even Mesopotamia. In these ancient civilizations, such philosophers as Pythagoras consumed themselves with establishing a quantifiable measure to explain the phenomenon of emitted sounds as well as the relationships between sounds (Boyer, 2011). This study of sounds is thriving and continues today in the fields of physics and music theory. What has been discovered by both physicists and music theorists is that there are universal patterns and relationships between musical elements that underpin our understanding and acceptance of organized sound as music (Mirelman, 2013). This unit will examine the structural organization of sounds which are called musical form and will show how this organizational structure applies to the graphic organizational functions of rational mathematical expressions.

Before we get to the lessons, we need to establish a little background connectivity between music and math. I am sure you are wondering, "First, what is music theory and, second, what is musical form and how does it have anything to do with rational expressions?" To answer the first question, we will turn to David Fallows definition from the Oxford Companion to Music. He says the following:

> The term is used in three main ways in music, though all three are interrelated. The first is what is otherwise called "rudiments", currently taught as the elements of notation, of key signatures, of time signatures, of rhythmic notation, and so on. Theory in this sense is treated as the necessary preliminary to the study of harmony, counterpoint, and form. The second is the study of writings about music from ancient times onwards. [...] The third is an area of current musicological study that seeks to define processes and general principles in music — a sphere of research that can be distinguished from analysis in that it takes as its starting-point not the individual work or performance but the fundamental materials from which it is built (Fallows, 2017).

Essentially, music theory is philosophical practice which studies the possibilities as well as applications of music and develops analytical apparatuses to systematically evaluate the sounds. Just like mathematical proofs, music theory techniques are developed to prove that something is possible and describe all the possible behaviors of a specific sound configuration. Music theorists spend their time interrogating and arranging the means that composers and musicians employ to create pleasing sounds. The variables that are analyzed include things like voice-leading, tuning, consonance, dissonance, resolution, rhythmic relationships, orchestration, ornamentation, improvisation and even electronic sound emission. To study these musical elements, methods of analysis have been developed over the years that are comparative, descriptive, statistical, and more. These methods employ traditional mathematics as well as the frequent use of Western music notation dissection and graphic analysis to address and evaluate things like musical acoustics, musical notational necessities, and tonal composition techniques like harmony and counterpoint.

To answer the second question, what is musical form and how does it have anything to do with rational expressions? we need to define musical form. Peter Scholes defines musical form in the tenth edition of The Oxford Companion to Music as "a series of strategies designed to find a successful mean between the opposite extremes of unrelieved repetition and unrelieved alteration." (Scholos,1977). Another way to think of musical form is as a general organizational or representation of ideas in the order in which they were musically executed. A properly delineated musical form is a coherent representation of musical idea statements, repetitions, and/or diversions such that the connections between contrasts and similarities are obvious. The ability to extrapolate from musical language these organizational patterns is crucial to understanding the behaviors of a piece of music and ultimately to understanding a composer's overarching musical goals. The employment of formal diversity in music has many uses in manipulating the effectiveness of the musical material at conveying certain feelings and emotions in the listener (Latham, 2002). Understanding the form of the music that we listen to helps us to understand the implications of certain sounds and gives us the tools that can be helpful in categorizing and comprehending the behavior of the music and how it corresponds to our reactions to what we are hearing. Musical form provides the listener with a map of possibilities through which they can filter their expectations and interpret what they are hearing. These interpretations can be as simple as, "this must be the chorus because I heard this music before with the exact same words" or as complex as "this must be the development of the second movement of the sonata because the themes are speeding up and the harmonic motion is moving through different related keys." Formula helps us to bring order and comprehension to the observed behavior of our sonic environment so we can hear the content relationships between sounds.

Consider this, what is a formula? For the purposes of this unit, I will define a formula as a statement that is usually represented in symbols and serves the purpose of establishing an organizational paradigm for use in the execution of a desired procedure. In math, this is expressed in combinations of algebraic symbols and in music it is expressed in combinations of letters represented as notes. Formulas are representations of content behavior in both math and music. Just as musical form describes the sequence of musical ideas as heard by the listener, mathematical formulas define the visual representation of an expression as seen on the graph. Formulas are a means to organize mathematical definitions which create a function that, when graphed, describes mathematical relationships. For instance, the graph of the function $y=mx+b$ is a straight line so any graph of a straight line can be reduced to a formula in slope intercept form. The same is true for music. A song in ternary form has a predetermined organization of musical ideas that is ABA. As such, if you hear a piece of music that has the same theme at the beginning and the end with something completely different in the middle, the song is in ternary form, also known as ABA form. Also, if a phrase’s beginning is written with an ending that rises from the home note like the end of a spoken question, the antecedent-consequent phrase formula describes to us that what will follow is to be a phrase ending which returns to the home note for that piece. This musical phrase behavior is dictated by a musical formula just like a graph's behavior is dictated by a mathematical formula.

Mathematical formulas have patterns not unlike music. The simplest family of functions are the linear functions. Linear functions include the constant functions (same output for every input), the functions that increase at a constant (positive) rate, and the functions that decrease at a constant (negative) rate. Words and phrases like "constant," "constant rate," "steady," "regular," and "equal change over equal intervals" are all features of the linear family. Algebraically, linear functions are a two-parameter function. In the formula $y=bx + a$ the parameter a is called the function's y-intercept and the parameter b is called the slope. Together, they completely determine a linear function's input-output behavior. If $a = 0$ then the function reduces to $y = bx$ and the y intercept a is zero. The function simplifies to $y = bx$, or a proportional relationship between “y” and “x” with “b” as the constant of proportionality.

When $b = 0$ the equation simplifies to $y=a$ and the function is the identity as y is a constant. It is important for student to understand the impact of the a and b variable roles when each equal zero.

The quadratic equation is like the linear equation as it has variables for each term in the equation. The standard equation is $y = ax^2 + bx + c$. Like the linear function, a quadratic function has a quadratic portion ax^2, a linear portion bx and a constant portion c. The three parts are comparable to the tertiary form in music as third parts define the equation.

Objectives:

This unit is designed to connect high school course content in math to real world happenings using music concepts. Specifically, the unit is for high school mathematics students enrolled in some math course meeting five days per week for fifty-five-minute periods. Students frequently learn facts and figures, and feel they are not relevant to their lives. This unit will connect students to math through the music for which they already have an affinity. Students will be able to apply math that they learn in class to solve real world problems and create meaning for the relationship forged. This unit will emphasize the importance of education across two disciplines: music and mathematics. Students will use the different forms of music as models for the formulas and procedures learned in math. This experience will impact their academic exposure and educational outlook.

The overall objectives of the unit are:

- To enable students to construct knowledge that the music they listen to has a pattern or form just as mathematics has patterns and formulas
- To teach students that different types of music have their own forms which are distinct and are used to define that specific type of music just as math models for the formulas and procedures as distinct when learning math

- To solidify student understanding of the families of functions, their graphs, characteristics and formulas.

Strategies

The strategies of the unit will be consistent with the objectives outlined above and will include music of different genres but a focus on the music that students find interesting. They will learn that music has patterns and then use patterns to solve math problems following a procedure that much like to music has patterns in the procedures utilized.

Strategies will incorporate the eight "Standards of Mathematical Practices":

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning

Classroom Activities

Lesson 1: A Survey of Music Form – How to Hear the Essential Elements of Form

Objective: To learn the definition of musical form

Specific Outcomes: SWBAT isolate the melody and the rhythm for a song
SWBAT recognize A, AB and ABA forms of music

Materials: Computer with internet access; video projector, video segments of songs identified in the lessons (video explanations of form), screen or smartboard

Procedure Class 1

Using the computer and video projector and video segments, introduce the concept of musical form. Start with the YouTube video Form and Structure Video found at:

https://www.youtube.com/watch?v=D8j8bYeo3Wk

Continue with this video that introduces form in specific terms. This is a video which describes musical form and describes AB form in detail using three different popular music segments: Jiggle Bells, Happy, Star Wars theme.

https://www.youtube.com/watch?v=T5wTqFteQVY.

For homework, students should write a paragraph describing how music informs the public about issues in society and how opinions are formed on those issues because of the music.

Procedure Class 2

The ABA form is introduced using Twinkle Twinkle Little Star in this second you tube video: https://www.youtube.com/watch?v=ErqqwHJtdWI

Play this video, then play a recording of the song. Reinforce the A and B parts by having the students say A when the first part plays, B when they hear the second section and A when the third section returns.

Follow-up: The end of the video invites students to listen for the ABA form in their own songs. Students should find songs with the ABA form and bring them to class on their phones to play in class. Students should listen to the song then listen again and identify the A the B and the A as each song plays.

Lesson 2: Discovering a Family of Functions

Objective: To learn four basic families of functions by graphing data

Specific objectives: SWBAT graph data and draw conclusions concerning the nature of the functions; SWBAT identify and differentiate each function family based on their graph.

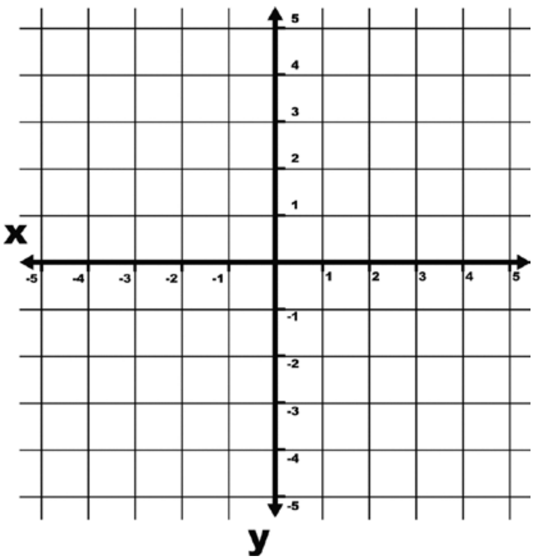
Materials: Data sheet containing raw data for student graphs, graphing calculator, graph paper, pencil.

Procedure:

1. Working in pairs, students will plot the data from each table on the graph paper provided (Each data set should be graphed separately on graph paper square provided.)
2. Using the graphing calculator, students will use the Stat, list function and enter the data into the TI-84 series calculator.
3. Each data set will be graphed on the calculator to confirm the hand-written results and determine the equation of the line.
4. Using the student worksheet, describe the characteristics of the graph in comparison to one another.
5. Students will predict the type of graph: constant, linear and quadratic

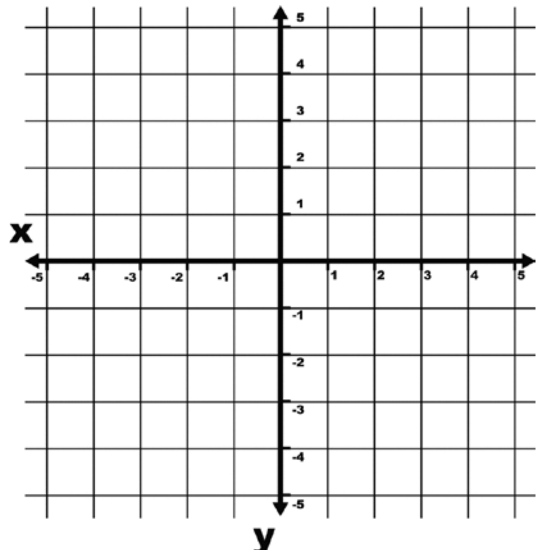
Graphing Family of Functions

Graph A



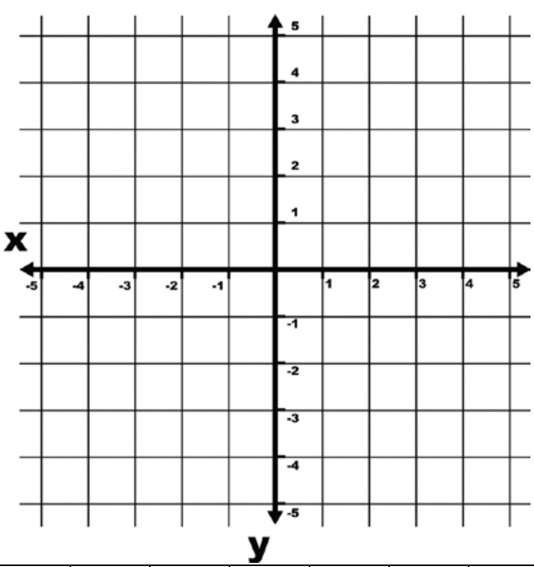
x	-2	-1	0	1	2	3
y	-2	-1	0	1	2	3

Graph B



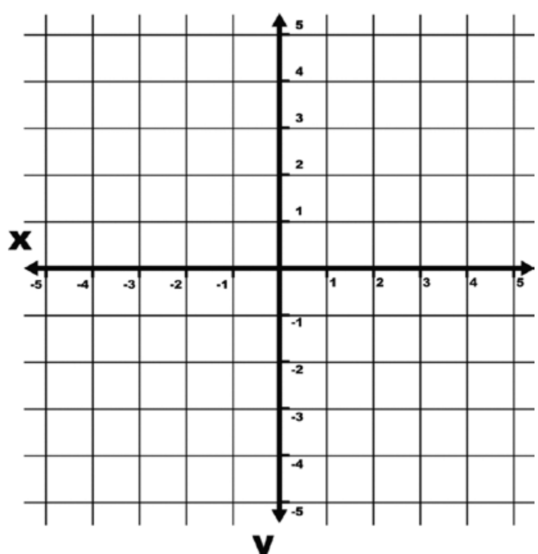
X	-2	-1	0	1	2	
y	4	1	0	1	4	

Graph C



x	-2	-1	0	1	2	3
y	4	4	4	4	4	4

Graph D



x	-2	-1	0	1	2	3
y	-4	-3	-2	-1	0	1

Lesson 3: Putting Music Form into Math Formulas

Objective: To combine the music concept of form and the algebraic concept of math formulas.

Specific Objective: SWBAT connect the concept of musical parts which make up a form with the parts of a function and the similarities across function families

Materials:

Student graphs from lesson 2; musical forms definitions (lesson 1)

Procedure:

1. Review the musical forms discussed in lesson 1 with students. Emphasize the three musical forms: A, AB and ABA (teach rondo ABCA)
2. Students should review the equations for constant, linear and quadratic functions (y = b, y=x; y=mx+b, y = ax^2+ bx + c.
3. Students sho9uld explain the variables and coefficients for each equation and the representative parts.
4. Students should complete the table of formulas for each mathematical formula graphed in lesson 2 (see below).

Graph	Equation	Rule	Music Form	Musical Parts
A				
B				
C				
D				

5. Match each graph with a musical form in the table above.

6. Explain how you matched each of the four graphs to its musical form

Graph A: ____

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Graph B: ____

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Graph C: ____

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Graph D: ____

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Fallows, David. "Theory". *The Oxford Companion to Music*. Oxford Music Online. Retrieved 20 May 2017.

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Mirelman, Sam (2013). "Tuning Procedures in Ancient Iraq". *Analytical Approaches to World Music* 2, no. 2:43–56.

Pickover, Clifford A. *The Math Book from Pythagoras to the 57th Dimension, 250 Milestones in the History of Mathematics*. Sterling: New York, NY, 2009. Print.

Scholes, Percy A. (1977). "Form". *The Oxford Companion to Music* (10 ed.). Oxford University Press.

Teacher Resources

Schultz, Ellis and Hollowell, Engelbrecht (2003). *Annotated Teachers Edition 2.* Holt Rinehart Winston: New York, NY, 2003. Print
This textbook contains math content including functions, parent functions, families of functions and all the individual functions which make up each family.

Music in the Classroom: Effects on Math Anxiety and Mathematics Achievement Author: Heimerman, Jacob T Citation: Newsweek, 19/02/1996, Volume 127, Issue 8, p. 54 Benefits of music and mathematics on student learning

The Impact of Music on Student Achievement in the Third and Fifth Grade Math Curriculum Author: Albright, Ruth E Citation: 1/1/2012

Student Resources

Loy, Gareth Musimathics: The Mathematical Foundations of Music
Citation: 6/06/2006
This reference provides a mathematical foundation for music for students.

http://mathdemos.org/mathdemos/family_of_functions/famfuncs.html
This website contains a group of animations grouped together which provide a student tool for studying families of functions. Each animation illustrates how functions changes when certain parameters changes value i.e. how the graph of a linear function changes as m changes value.

Appendix/Content Standards

Anchor Descriptor	Core Standard
A1.1.2.1 Write, solve and /or graph equations using various methods	CC.2.1.HS.F3Apply quantitative reasoning to choose and interpret units and scales in formulas, graphs and data displays CC.2.1.HS.F.4 Use units as a way to understand problems and to guide the solution of multi-step problems. CC.2.2.8.B.3 Analyze and solve equations CC.2.2.8.C.1 Define, evaluate, and compare functions. CC.2.2.8.C.2 Use concepts of functions to model relationships between quantities. CC.2.2.HS.C.3 Write functions or sequences that model relationships between two quantities. CC.2.2.HS.D.7 Create and graph equations …to describe numbers or relationships.
A1.2.1.1 Analyze and/or use patterns or relations.	CC.2.2.HS.C.2 Graph and analyze functions and use their properties to make connections between the different representations. CC.2.2.HS.C.3 Write functions or sequences that model relationships between two quantities.
A1.2.2.1 Describe, compute, and/or use models to describe functions	CC.2.2.HS.C.2 Graph and analyze functions and use their properties to make connections between the different representations. CC.2.2.HS.C.3 Write functions or sequences that model relationships between two quantities. CC.2.2.HS.C.5 Construct and compare linear, quadratic, and exponential models to solve problems. CC.2.2.HS.C.6 Interpret functions in terms of the situations they model. CC.2.4.HS.B.1 Summarize, represent, and interpret data on a single count or measurement variable.

MUSIC

Big Idea	EQ	Concept	Competency	Standards
The skills, techniques, elements and principles of the arts can be learned, studied,	Why it is important to be able to create, recreate and perform	While much of the school-based musical experience happens within a group, it is also important for	Demonstrate the ability to independently create, recreate, rehearse and perform musical works and explain	9.1.12.A, Know and use the elements and principles of the music art form to create works in the arts. 9.1.12.C, Integrate and apply advanced vocabulary to the arts forms

Big Idea	EQ	Concept	Competency	Standards
refined and practiced.	music independently?	people to be able to create, recreate, rehearse and perform music independently.	why this is important.	9.1.12.H Evaluate the use and applications of materials to the arts.
The arts provide a medium to understand and exchange ideas.	As technology has changed, how has it changed the way we make music?	Contemporary technology allows people to share and collaborate on musical ideas.	Collaborate with others to create a musical work using contemporary technologies.	9.1.12.K Analyze and evaluate the use of traditional and contemporary technologies in furthering knowledge and understanding in the humanities
People have expressed experiences and ideas through the arts throughout time and across cultures.	How and why do works in the different arts disciplines share characteristics?	There are similarities between works in different arts disciplines from different time periods and different cultures.	Explain similarities between works in dance, music, theatre and visual arts in various cultural and historical contexts.	9.2.12.L Identify, explain and analyze common themes, forms and techniques from works in the arts