Race and Politics of Redistricting
Who Will Win?

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Overview

This curriculum unit will be created for use in a high school Mathematics classroom but could be used in a United States History or Civics classroom while the history of voting and the politics of voting rights are taught. As part of the processes of choosing leaders, Americans have long been inundated with political messages proclaiming the merits of candidates and policy positions. Legislation has passed in both the Congress and State governments that directly affect both adult and student's lives in Philadelphia. My students are unaware of their ability to have an impact on who governs or on the laws passed by elected officials. They frequently do not connect the real world with the material they are learning in school, particularly math and are heard to ask, "Will I ever use this?"

In fall 2012, the United States will elect a President. This unit will provide an understanding of the electoral process and expand their knowledge of how politicians in general and the president specifically are elected. They will also link the political process and math, probability and several mathematical models for analyzing votes. Students will be challenged to draw conclusions based on data they have collected in their school community and extrapolate to the Pennsylvania system for electing the President.

Rationale

The seminar, in which this unit was written, "American Racial Politics", examined American politics and the importance of race for politics and politics of race, from the mid-17th century through the upcoming 2012 elections. My unit will utilize the ideas from the seminar to enhance my students' historical understanding of race and the political process and the importance of the African-American vote. By utilizing real data, they will see the power of mathematics in voter pools, polls and the outcome of elections. This unit will emphasize the importance of education across many disciplines. My students frequently learn facts and figures, and feel they are not relevant to their lives. They will use real data to model and draw conclusions for real political activity. This will impact their lives and educational outlook.

Background

The seminar, “American Racial Politics.” examined race and American politics in many regards, including the role of race in election campaigns and voter behavior. This unit will consider the impact of districts and redistricting on election outcomes.

Section II of Article I of the US Constitution states, “The House of Representatives shall be composed of Members chosen every second year by the People of the several States …Representatives…shall be apportioned among the several States which may be included within the Union according to their respective Numbers.” While this provision was included in the Constitution, the framers did not dictate how it should be carried out. James Madison’s writing in the *Federalist Paper* Number 56 said the new nation should “divide the largest state into ten or twelve districts” but no proportional method was identified in the final document (Fair Vote 1724). Redistricting, the redrawing of geographical boundaries of congressional districts, is most often viewed as a way political parties chose to change or gerrymander the boundaries to retain or gain political control. As early as the 1820, census redistricting occurred for slave holder advantage. The fact that the Constitution granted extra representation to the slaveholding states via the infamous 3/5 clause in Article 1, Section 2 encouraged slaveholders to think that it was appropriate to structure their systems of congressional representation to bolster their “peculiar institution.” As Yale Law Professor Akhil Amar has written: “Unconstrained by any explicit intrastate equality norm in Article I and emboldened by the federal ratio, many slave states in the antebellum era skewed their congressional district maps in favor of slaveholding recipients within the state” (Amar 2005, 97). Then in response to concerns that some states were assigning too many of their congressional seats to certain powerful regional districts, the Apportionment Act of 1842 contained a requirement for single-member districts. It states the representative “should be elected by districts comprised of contiguous territory equal in number to the number of representatives to which said state may be entitled, no one district electing more than one representative.” This was intended to bring partisan fairness and to end at-large elections. (Fair Vote, 1724). Apportionment acts continued to affect American elections for many years particularly to enforce districting requirements as populations shifted and to insure contiguous district lines.

Today after each census, state legislatures draw new district maps for the election of members of the House of Representatives as well as the state legislatures. Often litigation arises which mean that legislative districting plans must be approved by the courts prior to implementation. In addition, some regions with histories of vote suppression are required to “pre-clear” their electoral rules, including districting, with the U. S. Department of Justice.

Those requirements exist because, although voting discrimination based on race was deemed illegal with the passing of the 15th Amendment and strengthened by the 1965 Voting Rights Act, racial minorities still face discrimination (Fair Vote, 830). While often not as overt and localized as poll taxes and literacy tests, voter-ID laws, purging voter rolls, providing insufficient polling booths and moving polling places serve the same outcome. The tactics of our current age couched as non-race conscious acts by proponents and discriminatory by opponents play as important a role as previous issues. The battleground of districts also extends to the creation of majority-minority districts. The politics of voting rights, long focused on voter exclusion, reconfigured significantly after 1965, into questions of vote dilution (King and Smith 2011, 172-173). Race-conscious redistricting is therefore a focus in shaping the outcomes of elections as well. There is a distinct correlation between the percentage of minority voters in a district and the party affiliation of the legislator elected: the higher the percentage of minority voters, the greater the probability of electing a Democrat to office. If the district is a majority-minority district, there is a high probability that an African-American or Hispanic Democrat will be elected (Grofman 2003, 44-45).

As a result, Republicans in many states adopted new laws when in power during the 2000s and especially after their successes in the 2010 election, to improve their electoral prospects in 2012. The Republican-controlled state legislature of Pennsylvania adopted a districting plan that survived judicial challenges and will likely ultimately be decided by the Supreme Court. It also enacted a controversial new voter-ID law (Infield 2012; Associate Press, 2012: Couloumbis and Ritger, 2012). The new districting arrangements and the voter-ID law will likely shape current events in this battle ground state for the 2012 Presidential election. Students will explore these concepts as they vote and analyze the ballot results.

To prepare them to do so, this unit provides them with an understanding of probability (the mathematical theory of chance), different elements of mathematics involved in estimating voting probabilities, a key to judging the likely electoral consequences of different districting arrangements; the consequences of different systems of vote counting; and the impacts of districts themselves. To make the mathematics involved more accessible to students, familiar scenarios are utilized to introduce chance then hypothetical voting problems and systems are used. The skills developed through these lessons can then be applied to real-world voting and districting issues.

Objectives

This curriculum unit is designed for high school math classes. It is most appropriate for Algebra but could also be used in Discrete Mathematics and Statistics courses. The Pennsylvania State Standards for both Algebra and Social Studies (Civics and Government) are indicated in the unit. Students will be exposed to historical information concerning the racial politics concerning voting and voter disenfranchisement while studying mathematics. Informational texts are one of the instructional focuses next year, therefore, the readings for history and math tie-in with this new Core Curriculum Standard. This year, we are also seeking opportunities for multidisciplinary applications across the curriculum. There are rich opportunities for math and social studies joint lessons in the history and operation s of voting districting systems.

The curriculum unit will be used in my classroom to make real connections to probability, voting methods and the politics of redistricting. My classes will learn about the constitutional provision for electing congressional representatives. Students will develop hypotheses on how the outcome of an election is likely to change based on what the structure of the electoral process is, and they will consider how districting and laws affecting access to voting could affect the upcoming presidential election. Students will learn about congressional districts, examine the 2010 district maps and compare it to the 2012 proposed revision. They will hypothesize on why the changes were made and what effect the changes could have on the outcome of elections. They will also consider the outcome in Philadelphia of the voter-ID law and its projected effect on voting. My unit will be used to enhance the teaching of probability in the other mathematics classrooms in my school by making it possible for them to use voting data.

Standards

Pennsylvania Academic Standards

MATHEMATICS

2.2 A Develop and use computation and concepts, operations and procedures with real numbers in problem-solving situations

2.6 B Explain effects of sampling procedures and missing or incorrect information on reliability

2.7 D Use experimental and theoretical probability distributions to make judgments about the likelihood of various outcomes in uncertain situations

2.7 E Solve problems involving independent simple and compound events

2.7.8 E Make valid inferences, predictions and arguments based on probability

SOCIAL STUDIES

5.3.12 A Analyze and evaluate the structure, organization and operation of the local, state, and national governments including domestic and national policy-making.
5.3.12 F Evaluate the elements of the election process.

Classroom Activities

Lesson Plan 1- Probability

Description: In this lesson, students will create data, determine sample space and determine theoretical probabilities using their preferences for Italian water ice.

Learning Objectives

Students will:

- Determine sample space for problems of chance.
- Find the theoretical probability of an event
- Use probabilities to predict outcomes

Materials

- Worksheet 1 - Italian Ice Flavor Preference Sheet
- Chart paper (6-8 sheets)
- Colored markers (water based)
- Italian Ice Flavor Ballot
- Worksheet 2 – Class Ranking Results
- Ranking sheet (student and summary)

Classroom arrangement: Students should be sitting in groups of 4-6 people (teacher should count the number of students sitting in the class).

Begin class by showing students menu or selection of items. Tell students they will be collecting data today and exploring probabilities. Begin the lesson with a discussion of theoretical probability. Write the formula for probability on the board:

$$P(\text{Event}) = \frac{\text{Number of Favorable Outcomes}}{\text{Total Number of Outcomes}}$$

Invite each group to count the number of students who are male and female in their group. Create a chart on the board to record the answers:

Group	Male	Female
Group 1		
Group 2		
Group 3		
Group 4		
Total		
Class Total Students		

Model finding the probability of being male or female in this your class: Add up the number of female and the number of male students. Then using the formula, determine the theoretical probability of being male or female in this class (number of male students divided by total number of students); repeat for female students. Reduce the fraction (modeling how) and then determine the decimal to the nearest hundredth. Ask if this is the same in another teacher's class down the hall. Note that just as the probabilities of being male or female may vary in different classrooms, even though the probabilities for the people as a whole remain the same, the probabilities of having Republican or Democratic voters can vary in different districts.

Provide each group with a simple probability question and have them report out on chart paper. Encourage creative display of on their chart paper

Example question:

Zaire has a bag of candy. Each piece of candy is a different color. There are 9 red, six yellow, 6 green, 3 orange, 12 purple candies in the bag. Find the probability of selecting a yellow candy if you randomly select a candy from the bag. Find the probabilities for the remaining colors in the bag.

Probability Terminology (Holt Algebra 2 p. 628):

Term	Definition and Example
Probability	Overall likelihood that an event will happen (value zero – one)
Event	An individual outcome; any specific combination of outcomes (picking red candy from bag)
Outcome	Result
Random Outcome	All possible outcomes are equally likely
Trial	A systematic opportunity for an event to occur (selecting out of a bag)

Experiment	One or more trials
Sample Space	The set of all possible outcomes (all the items)

Distribute worksheet 1 - Italian Ice Flavor Preference Sheet to each student. Students should circle the 10 flavors that they like; then rank the 10 flavor choices 1-10 with 1 being their favorite flavor. Distribute the ballots. Students should write their number one flavor on the bottom of the worksheet and on the "ballot". Ballots are collected and tallied by flavor and a winning flavor is announced. The results are listed and the probability of liking a flavor is calculated using the probability formula.

Next distribute the flavor ranking sheets. Students should transfer their top four flavor preferences onto the ranking sheet. Original flavor preference sheet and ranking sheet should be collected. The sheets must now be tallied by flavor order. Sort the sheets so that identical rankings are together (e.g. all sheets with 1st choice - Lemon, 2nd choice - Cherry, 3rd choice - Mango, 4th choice - Green apple are counted and placed in one pile). Complete the ranking summary chart (this may take a while – start the next lesson; use this data at the end). The ranking chart may represent all classes rather than one class.

Usually two or three flavors will prove to be most popular and therefore most probable. Teachers might note that although in most American elections, there are two major party candidates who get most of the attention, there are often also other candidates on the ballot. They are rarely popular political flavors; but in close elections, the few votes they take away from one of the major party candidates may lead to the election of the other. Most observers believe, for example, that Ralph Nader caused Albert Gore to lose the 2000 presidential election to George W. Bush by taking hundreds of thousands of votes away from him in swing states.

See Appendix for worksheets

Lesson 2 The Plurality, Borda Methods and Other Voting Systems
(Source: NCTM Lesson Will the Best Candidate Win?," © 2008 National Council of Teachers of Mathematics)

The overall winning flavor of Italian water ice for our class was determined using the plurality method. We use this method for many voting processes in America, though it is less common in democratic systems around the world. In many parliamentary systems, parties win seats on the basis of their proportion of the overall vote, rather than getting seats only in districts in which their candidates received the most votes. In this lesson we will consider different voting results based on various voting methods.

This lesson plan for grades 9 - 12 is adapted from an article in the January 2000 edition of Mathematics Teacher entitled "Will the Best Candidate Win?" It describes the following activities that allow students to explore alternative voting methods. Students discover what advantages and disadvantages each method offers and also see that each fails, in some way, to satisfy some properties widely considered desirable.

Learning Objectives

By the end of this lesson, students will:

- See connections between mathematics and other disciplines, including political science, history, ethics, and sports;
- Develop skills in mathematical reasoning and apply those skills to everyday situations, including elections.
- Learn about various voting methods, ways in which these methods can be manipulated to achieve certain outcomes, and the impossibility of inarguable fair elections when more than two alternatives are available

Materials

- Activity Sheet 1: The Plurality Method and Other Voting Systems
- Activity Sheet 2: Strategic Voting
- Activity Sheet 3: Tournament Digraphs and Condorcet Winners

Instructional Plan

Students think they are familiar with the concept of voting - after all, they have heard about governmental elections, Academy Award voting, and the ranking of the sports teams. They may have also participated in club and school elections. Yet if you ask students about voting methods, most can describe only one method: plurality. Using this method, the option that gets the most votes wins, by whatever margin. But is it really fair when, for example, a candidate is elected even though a majority of voters preferred someone else—and even though, if their second choices were taken into account, a different candidate might be shown to have the broadest support? Most students do not realize this can and does happen. They have never questioned the fairness of plurality, and they have never considered alternative voting methods.

The following activities allow students to explore alternative voting methods, including many that are actually used in democratic systems in other parts of the world. Students discover what advantages and disadvantages each method offers and also see that each method fails, in some way, to satisfy some properties we might wish them to have. Many students are particularly surprised to discover that when the plurality method is used, the winner could be the candidate who the majority of voters like the least.

In addition, students look at how elections can be manipulated. One extension involves discussing economist Kenneth Arrow's impossibility theorem. It states that when voters are presented with more than two options, it is impossible for a voting system to satisfy all the features most people consider desirable. Although these activities, including the statement and proof of Arrow's theorem, require only basic arithmetic, they allow students to engage in high-level mathematical thinking.

This activity lends itself easily to interdisciplinary instruction. Current events on the national, local or school level can be incorporated into the project. In addition to elections, if students are involved in making group decisions on such things as, for example, choosing a class gift or service project, selecting a time to hold an event, or arranging for refreshments, teachers may want to substitute relevant activities for those on the worksheets.

To introduce the topic and to familiarize students with this table format, teachers may want to start with the following whole-class activity:

Have students suggest activities or destinations for a hypothetical class trip, and write the first 3 suggestions on the board. Ask each student to list her or his first, second, and third choices on a piece of paper, permitting no tied rankings.

Understanding the Table

Ask two different students for their preferences. Ask the class to help generate all 6 possible preference combinations. Suppose your students suggested archery (A), biking (B), and canoeing (C). You would then create a table, similar to the one below, where the columns represent all the possible preference lists.

Example Preference Table						
First Choice	A	A	B	B	C	C
Second Choice	B	C	A	C	A	B
Third Choice	C	B	C	A	B	A

Tallying the Class's Preferences

At the top of each column, write the number of students who ranked the options in the order given. Ask students which alternative wins and how they determined the winner. If students generate only 1 method of tallying the winner, ask them to think of a different way.

Use the activity sheets below to help students explore different voting methods. Each activity sheet is designed for groups of 3 or 4 students.

Practice with the Plurality Method and Other Voting Systems

Activity Sheet - Plurality Method and Other Voting Systems (*see appendix for worksheet*)

Plurality voting is the method most familiar to students. In this method, each voter is given 1 vote and the option that receives the most votes' wins. Variations of the plurality method are used in choosing state representatives and senators, ratifying proposals, and selecting Academy

Award winners. When a candidate receives more than 50% of the vote, including situations in which there are only 2 candidates, then the plurality method does produce a preferred candidate. However, in many situations, plurality may not produce a clear preference. Remind students that there are often more than two candidates on the ballot, especially for the Presidency, and those candidates can be and have been elected even though a majority of the voters opposed them. Examples include Abraham Lincoln, Richard Nixon, Bill Clinton, and George W. Bush.

Since students are comfortable with the plurality method, they can usually complete this activity sheet (*in appendix*) in small groups. Some of the questions may provide especially good topics for class discussions:

Question 3: Students are often surprised to find that skiing comes in both first and last.

Question 4: A reasonable answer is the speed and ease of the plurality method.

Question 5: If students get stuck, ask them to think how winners are determined in sports tournaments or when many alternatives are available.

The second half of this activity sheet introduces students to 3 widely used voting methods: the Hare system, Borda count, and sequential pair-wise voting. Although the sheet gives instructions for each voting method, some student groups may need help following the directions.

Question 7: With the Borda count, show students how they can put a 0 next to third place, a 1 next to second place, and a 2 next to first place on the chart as an aid to totaling the points. A quick way for students to verify that their totals are reasonable is to determine the total number of votes - that is, 3 points per voter times 40 voters, and verify that this result matches their total number of points.

Question 8: Comparing the technique of sequential pairwise voting with a single elimination tournament with byes, may help students understand the method, but care should be taken to distinguish between the two. In sequential pairwise voting, the pairings are sequential and no simultaneous pairings take place. Sketching the corresponding "tournament bracket" diagrams may help clarify the distinction.

Notes:

A nice follow-up to this activity sheet is a discussion about variations of the plurality method, including runoff elections, the electoral college, and the two-thirds majority required for Congress to send constitutional amendments to the states for ratification (and three-fourths of the states must then approve).

Another variation of plurality, the Hare system of voting, involves a series of elections. At each stage the option or options with the least number of votes are eliminated from future ballots. The voters who originally voted for the eliminated option vote next for the remaining option that they have ranked highest. This point is worth emphasizing because students tend to overlook these voters. The Hare system allows them still to have influence in the election even though their first choice candidates lose.

Ties do not occur in the problems given; however, you should know how they are handled in the multi-step voting systems. With the Hare system, whenever 2 or more options share the least number of votes, they are eliminated at the same time. If all remaining candidates have the same number of votes, none are eliminated; they are all considered tied for the win. Whenever 2 candidates tie during a head-to-head contest in sequential pair-wise voting, neither is eliminated; they both continue and compete in a 3-way contest with the next candidate.

Have students discuss the advantages and disadvantages of each of these voting methods, first in their small groups and then with the entire class. If students can benefit from more practice, teachers may reinforce the voting methods using the previous Italian Ice rank summary.

Lesson 3 Voting and Districts/Redistricting

Description: In this lesson, students will vote on class representatives first within their own class group, then across districts, after "redistricting" and at large voting. Students will analyze the impact on the candidates elected. (Students may also prepare a school-wide voting process based on a topic of interest or student council candidates, vote by advisory sub-group, by grade and the whole school).

Learning Objectives

Students will:

- Experience the impact of constituent grouping on voting outcomes
- Predict the outcomes of elections within their small communities and draw conclusions for US elections

Materials

- Class roster
- Ballots (student created)
- Calculator
-

Procedure

In this exercise initially a teacher's class roster represents a state in the US. Then each class becomes a district and all classes combined become the state.

1. Take the names on one class roster, in alphabetical order, and divide them into groups of 4-6 students (include all students – even those chronically absent).
2. Create a "map" of the groups (seating chart)

3. Inform students that they will be electing group representatives and have a discussion to determine the criteria for the ideal candidate
4. Seat students in the groups created from the alphabetical roster and share the “map” with all students
5. Remind students of the criteria of a good candidate and task each group with electing their own candidate.
6. Alter the map so that original groups become districts (keep all boundaries, just combine them). Hold an election for one candidate in the new district (determine rules for a runoff ahead of time in case of a tie)
7. It’s Census time! Look at each group and drop chronically absent students from the district. Create districts having the same number of students (a redistricting plan has been created)

   How do students feel about the new districts? What do they think will happen in the next election?

8. Circulate new district maps (the new map should have fewer districts – ideally create lopsided groups so that smaller groups are added to larger groups creating disenfranchisement within some original groups)

   What changes would each group recommend to the new district map? Why? How does it feel to be a member of a new district? What if your original district was primarily Democratic and your new group is primarily Republican. How would you feel? Will the candidate that you elected last time win now?

9. Now take the winner from the new districts and have the class select a subset of the number of districts (vote for 3 out of 5 current winners). Explain that this is an at-large election. The entire class will vote.

   How did the at-large election affect the outcome? What if the at-large election were held across all of the teachers’ classes? Would the outcomes change? How would candidates be elected? What would a candidate need to do to educate voters on their positions?

10. Nominate new candidates for an at large election across all class. Discuss the outcome. Was it what students expected? How do at large elections effect representation? If each district had a different ethnic composition, how would the election outcomes effect representation?

This process can be extended across the entire school. The data collected could compare voting within advisories for student council representatives, across grade level for officers and across the school for school representatives. How does each process change the outcome?

Annotated Bibliography

Amar, Akhil Reed. 2005. *American's Constitution A Biography*. Random House: New York
Historical information on the American Constitution, district creation and voting rules for slave states and non-slave states, among many other topics

Associated Press. 2012. "Pennsylvania Senate GOP leader say no change in district lines for primary election," *Philadelphia Inquirer*, February 22, http://www.pennlive.com/midstate/index.ssf/2012/02/pennsylvania_senate_gop_leader.html.

Couloumbis, Angela and Ritger, Clara. 2012. "Dispute over Pennsylvania Voter ID Law Intensifies." *Philadelphia Inquirer*, July 18, http://articles.philly.com/2012-07-18/news/32731283_1_penndot-id-voter-id-law-voter-fraud.

Fair Vote: The Center for Voting and Democracy, http://www.fairvote.org. This website has a wide range of information on American voting and voting reform efforts, especially in its archives (http://archive.fairvote.org).

Grofman, Bernard, ed. 2003. *Race and Redistricting in the 1990's*. Agathon Press, New York, NY. A compilation of leading political science analyses of redistricting in the wake of amendments to the Voting Rights Act that encouraged minority-majority districts.

Infield, Tom. 2012. "Court Rejects Appeal in Redistricting." *Philadelphia Inquirer*, February 8, http://articles.philly.com/2012-02-08/news/31038048_1_election-districts-district-maps-legislative-reapportionment-commission.

King, Desmond S. and Smith, Rogers M. 2011. *Still a House Divided: Race and Politics in Obama's America*. Princeton University Press: Princeton, NJ, 2011, 168-191. An analysis of redistricting controversies as shaped by advocates of "color-blind" and "race-conscious" policies.

Teacher Resources

Crisler, Nancy. Fisher, Patience. Froelich. Discrete Mathematics Through Applications, Second Edition. W. H. Freeman and Company, New York. 1994
An excellent source of voter methods (Chapters 1 and 2) and counting and probability (Chapter 6)

Schultz, James. Wade, Ellis, JR. Hollowell, Kathleen. Kennedy, Paul. Algebra 2. Holt, Rinehart and Winston, 2003
An excellent source for problems, illustrations and examples for teaching mathematics content

http://www.pbs.org/newshour/thenews/thegov/story.php?id=19299&package_id=634

“Redistricting: Drawing the Lines” Social Studies Lesson Plan
www.pbs.org/newshour/dotnews/RedistrictingSSFinal.pdf Jan 20, 2012 ... Social Studies Lesson Plan is a feature of ... gerrymandering can affect the outcome of state elections and federal policy.

Mother Jones
http://motherjones.com/politics/2011/09/gop-electoral-college-plan-beat-obama-2012
The GOP’s Genius Plan to Beat Obama in 2012
Reviews GOP strategy for defeating President Obama

National Council of Teachers of Mathematics. 2008. “Will the Best Candidate Win,” © 2008 National Council of Teachers of Mathematics, http://illuminations.nctm.org.

Appendix
Worksheets for Lesson 1

Who Will Win? Worksheet 1

Name: ____________________________

Date: ____________________________

Italian Ice Flavor Preference Sheet

Find your favorite using the list below:

❶ Read through the list of flavors. Circle the 10 flavors that you like the most. (if you don't have 10 flavors that you like, just select the ones that you do like)

❷ Rank the 10 flavors that you circled in step 1, from favorite to least favorite. Place a 1 next to the flavor you like the best, 2 next to the second favorite and so on until you don't have any flavors left.

Flavor	Rank	Flavor	Rank
Alex's Lemonade (Lemon)		Mango	
Banana		Mango-Orange	
Blueberry		Ocean Spray® Cranberry	
Blue Raspberry		Passion Fruit	
Bubblicious		Pina Colada	
Cantaloupe		Pineapple	
Cherry Chocolate		Raspberry	
Chocolate Chocolate Chip		Raspberry Lemonade	
Chocolate Peanut Butter		Root Beer	
Cotton Candy		Sour Patch Kids Red	
Florida Orange		Strawberry	
Georgia Peach		Strawberry Colada	
Green Apple		Strawberry Margarita	
Honeydew		Swedish Fish®	
Island Fusion		Tropical Punch	
Juicy Pear		Vanilla	
Key Lime		Watermelon	
Kiwi-Strawberry		Wild Black Cherry	

❸ Write the flavor you ranked number 1 below:

My number Italian water ice flavor is ____________________________________

Ballot

Italian Ice Flavor Ballot

My #1 Flavor is

____

Period: _____________________

Ranking Sheet (student copy)

RANKING	FLAVOR
FIRST CHOICE	
SECOND CHOICE	
THIRD CHOICE	
FOURTH CHOICE	

Ranking Summary (for alternate voting method ranking)

	Number of Voters					
RANKING	10	7	1	10	4	8
FIRST CHOICE						
SECOND CHOICE						
THIRD CHOICE						
FOURTH CHOICE						

Lesson 2
Voting - Activity Sheet 1 NAME ____________________________

DATE:_____________

The Plurality Method and Other Voting Systems

The forty members of your schools adventure club are trying to decide what type of trip to take. The chart shows how the club members rank the three options.

	Number of Voters					
RANKING	10	7	1	10	4	8
FIRST CHOICE	Skiing	Skiing	Rafting	Rafting	Caving	Caving
SECOND CHOICE	Rafting	Caving	Skiing	Caving	Skiing	Rafting
THIRD CHOICE	Caving	Rafting	Caving	Skiing	Rafting	Skiing

A common method of voting is called ***plurality***. In this system, each person casts one vote for a choice and the option with the option with the most votes wins.

1. On the basis of the chart, which activity is the winner under the plurality system? Why?

2. Which activity is liked least by the largest number of members? That is, which activity is ranked third by the greatest number of voters?

3. Why might the plurality method not produce results satisfactory to all voters?

4. Why do you think the plurality method is used most often?

5. Think of some variations of plurality voting or other voting techniques that might prove more satisfactory to the voters. Within your group, describe or develop at least two other vote-tallying methods that haven’t been discussed in class.

The ***Hare voting system*** involves taking an initial poll in which each person casts one vote for his or her favorite option. The option receiving the least number of first-place votes is eliminated, then another poll is taken. Those who originally voted for the eliminated option vote for their second choice. Continually eliminate the least popular option until a single winner emerges.

6. Using the table of votes from the Activity Club, which activity would the club choose? Describe the process as the options are eliminated.

The ***Borda Count*** is a voting method that takes into account each voter's first, second, and third choices. Each first-choice vote is awarded two points, each second-choice vote is awarded one point, and no point is awarded for a third choice. This way, each choice is assigned a point-value.

Example: For the Activity Club, skiing has seventeen first-choice votes and five second-choice votes, for a total of 2(17) + 1(15) = 39 points.

7. Determine the total number of points for the other two activities, showing your calculations. Which activity has the most points using this method?

Sequential pair-wise voting involves a sequence of head-to-head contests. First, the group votes on any one of two of the options and then the preferred option is matched with the next option, while the 'loser' is eliminated. Continue eliminating the less popular option of a pairing, until one remains.

8. Suppose a club-member suggests that they should first vote between skiing and caving, and then the winner of that voting goes up against rafting, which is the remaining option. Which activity is chosen by that method?

9. Which of these methods - plurality, Hare, Borda Count, or sequential pair-wise voting – is the fairest in this situation? Why? Which is the least fair? Why?

10. Suppose that your preference is rafting. Devise a voting system that would enable rafting to be chosen and that would be found fair to the other club members.

Solutions to Activity Sheet 1

The Plurality Method and Other Voting Systems

6. In the first round of voting, skiing gets seventeen votes, rafting gets eleven, and caving gets twelve. Thus, rafting is eliminated. In the follow-up election, skiing gains one of rafting's votes, for a total of eighteen, whereas caving gets its original twelve votes plus ten votes from rafting, for a total of twenty two. Skiing is now eliminated, leaving caving as the winner.

7. Caving gets 2(12) + 17, or 41, points. Rafting gets 2(11) + 18, or 40, points. Skiing gets 2(17) + 5, or 39, points. Caving wins using the Borda count.

8. In the first vote, skiing gets eighteen votes to caving's twenty-two. Thus, skiing is eliminated and caving meets rafting in a head-to-head contest. This time, caving gets nineteen votes, whereas rafting gets twenty-one. Rafting wins.

9. Answers may vary; however, many students will assert that rafting gets an unfair advantage in problem 3.

10. A hint may be needed here. One possible answer is to eliminate skiing, the activity that the greatest number of voters ranked last, and then to hold runoff election between the remaining options.

Strategic Voting

Use Activity Sheet 2 - Strategic Voting

Here students investigate how a voter or block of voters can influence the results of an election by submitting a ballot that does not represent their true preferences. Although the terminology is avoided on these sheets, each problem on this sheet demonstrates that a property known as *independence of irrelevant alternatives* (IIA) does not hold for these voting methods. In other words, a losing candidate can win the election without any voters having moved the new winner ahead of the original winner in their preference lists. They may have moved other, irrelevant, candidates above or below one of these two. As an example, consider the following chart of preference lists:

	Number		
Ranking	*2*	*3*	*4*
First choice	A	B	C
Second choice	B	A	A
Third choice	C	C	B

Candidate C wins using the plurality method. However, if the two voters represented by column 1 switched the positions of A and B in their preference list, B would win by plurality. Note that the order of B and C was not reversed. By moving B ahead of an irrelevant alternative, A, in 2 preference lists, B was able to win.

Question 1 on this sheet is adapted from *Introductory Graph Theory* (Chartrand 1985, 168).

Solutions to sheet 2

Strategic Voting

1. An editor who was voting according to his or her true preferences would probably rank his or her school first and Big City High second, or vice versa. In this problem though, students should discover that another strategy benefits the editor's school. By ranking his or her school's team first and not including Big City High among the top ten, the editor's school gets 9(9) + 10, or 91, points compared with Big City's 9(10) + 0, or 90, points. Give some credit to groups who create a tie, but point out to them that they need not include Big City High in the top ten.

2a. If the plurality method is used, A wins, with 48 percent of the vote compared with B's 28 percent and C's 24 percent.

2b. If the voters in this group ranked B ahead of C, then B would win instead of A.

2c. If the Hare system of voting is used, then C is eliminated first. In the next round, B wins with 52 percent of the vote. The last 10 percent of the voters would be most disappointed

with this result. If they submitted a ballot with the ranking C, A, B, then B would be eliminated in the first round and A would beat C.

Tournament Digraphs and Condorcet Winners

Use Activity Sheet 3 - Tournament Digraphs and Condorcet Winners

At first, one would expect that if a candidate called a Condorcet winner, could beat each of the other candidates in head-to-head contests, that candidate should win the election in which all candidates compete. Students are surprised to discover that this so-called Condorcet-winner criterion does not hold for the plurality method, Borda count, or Hare system. Ask students to explain why it *does* hold for sequential pairwise voting.

- **Question 1:** Tournament digraphs are used to help students visualize the results of pairwise voting. Most students expect candidate B to win in every method.
- **Question 2:** The solution is the digraph in question 1. Students discover that although B might turn out to be the winner under each of these voting methods, they can only be positive that B wins by using sequential pairwise voting.
- **Question 3:** The questions are tangential to the main topic. You may wish to omit it or use it as a take-home bonus question.

With n candidates,

$$\frac{n(n - 1)}{2}$$

arrows are involved. Some students will write this expression in the form $1+2+3+...+(n - 1)$. One way of deriving the first formula is by noting that n points exist and that each point has an arrow to or from each of the other n - 1 points. Since each arrow touches two points, the number of arrows is

$$\frac{n(n -1)}{2}$$

If students used patterns to discover the formula $1+2+3+...+ (n -1)$, you can show them that

$$\begin{aligned} 1+2+3+\cdots+(n-1) &= \frac{1+2+3+\cdots+(n-1)}{2} + \frac{(n-1)+(n-2)+(n-3)+\cdots+1}{2} \\ &= \frac{1}{2}\big((1+(n-1))+(2+(n-2))+(3+(n-3))+\cdots+((n-1)+1)\big) \\ &= \frac{1}{2}(n-1)(n) \end{aligned}$$

The Condorcet Winner Problem

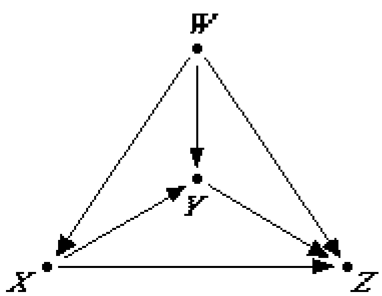
Tournament Digraphs and Condorcet Winners Activity Sheet also introduces the term Condorcet winner. Question 6 challenges students to be more creative and develop their own examples of tables of preference lists that show that the Hare system does not satisfy the Condorcet winner criterion. Make sure that students understand that their tables of preference lists for this problem must produce a Condorcet winner. Suggest that students think about how a Condorcet winner might lose an election under the Hare system of voting. Note that the Condorcet winner must be eliminated at an early stage; it cannot have a lot of first-place votes. Students can experiment later with different tables of preference lists to create this situation.

Solutions to sheet 3

Activity Sheet 3

Tournament Digraphs and Condorcet Winners

4. The tournament digraph follows:



The exact arrangement of the vertices W, X, Y, and Z is not important. To check students' graphs, verify that exactly one arrow appears between every pair of vertices, three arrows leave W, three arrows point toward Z, and an arrow goes from X to Y.

5. By using the Borda count, W gets 3(3) + 2(2) + 2 = 15 points, X gets 3(4) + 2(3) = 18 points, Y gets 2(2) + 7 = 11 points, and Z gets 3(2) + 2(2) = 10 points, so X wins. Have students verify that they have calculated the right number of total points for each option.

6. One example is given by the following table:

2	3	3
W	X	Y
X	W	W
Y	Y	X

Here, W is a Condorcet winner that gets eliminated in the first round of Hare voting.

7. Since sequential pairwise voting involves only head-to-head contests, a Condorcet winner will win every contest it is in and hence wins the election.

Questions for Students

1. Which voting systems are preferable if you are interested in both first- and second-place winners?
2. Which voting system is used in organized sports tournaments?
3. With which methods could "strategic voting" be effective?

Assessment Options

Extensions

1. **Arrow's Impossibility Theorem:** Now that students have discovered that each of these voting methods may not produce the expected result, a new question arises. *Does a voting system exist that satisfies all desirable criteria?*

   Kenneth Arrow proved that these 4 voting methods do not satisfy the Condorcet-winner criterion (CWC) or the IIA. He also proved that it is impossible to create any voting system that does. To be more precise, any voting system that always produces at least 1 winner cannot satisfy both CWC and IIA.

   The proof of this theorem, which can be found in Brams et al. (1996, 426-30), requires an understanding of the topics on these worksheets along with high-level mathematical reasoning. Students have difficulty understanding that if a criterion does not hold for any one table of preference lists under a particular voting method, then the voting method fails to satisfy the criterion. Other preference lists may exist for which there appears to be no conflict.

2. **Presidential Primaries:** Upcoming elections offer another opportunity for extending this activity and linking it with statistics and social studies. If more than 3 candidates are running for each party, have students survey a sample of adults, asking them to rank the leading Republican or Democratic candidates in order of preference. Ask students to create a table of preference lists illustrating their data, using each voting system studied. Students can report their sampling methods, calculations, and findings in an essay or news story.

Teacher Reflection

- How can you connect this lesson to other disciplines?
- Describe the level of understanding students had of each voting method based on the given definition and after the use of an example.
- Order the voting methods by amount of difficulty students had in developing an understanding of them.

- In what order will you present the voting methods when you teach this lesson in the future?
- What changes will you make to the activities to help students develop an understanding of each voting method?

NCTM Standards and Expectations

Data Analysis & Probability 9-12

- Understand how basic statistical techniques are used to monitor process characteristics in the workplace.

Number & Operations 9-12

- Use number-theory arguments to justify relationships involving whole numbers.

References

Written by Teresa D. Magnus. *Mathematics Teacher*, January 2000, page 18.

- Brams, Steven J., et al. "Social Choice: The Impossible Dream." In *For All Practical Purposes*, 4th ed., edited by COMAP, the Consortium for Mathematics and Its Applications, 411-42. New York: H. W. Freeman & Co., 1997.
- Chartrand, Gary. *Introductory Graph Theory*, New York: Dover publications, 1985.

Voting – Activity Sheet 2 NAME ______________________________

DATE:______________

Strategic Voting

Suppose that you are one of the ten sports editors whose votes together determine the rankings of state high school football teams. A variation of the Borda count is used in which each voter ranks his or her top ten teams out of the many teams in the division. Points are awarded as follows:

10 points for each first-place vote
9 points for each second-place vote
8 points for each third-place vote
7 points for each fourth-place vote
(continued until 10^{th} place votes are awarded)

1 point for each tenth-place vote

1. You suspect that the nine other sports editors will rank Big City High School first and your high school's team second. Show a ranking for each of the sports writers that will enable your team to earn the most number of points? Explain.

2. Candidate A, B, and C are running for office. The preferences of the voting community are given in the following chart:

PERCENT				
RANKING	**38%**	**28%**	**24%**	**10%**
FIRST CHOICE	A	B	C	A
SECOND CHOICE	B	A	B	C
THIRD CHOICE	C	C	A	B

a. Who will win the election if the plurality method is used?

b. By still using the plurality method, does a strategy exist that the voters in the 24 percent column could use to get more to their liking? How did you decide?

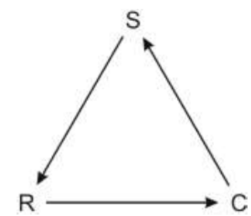
c. Who will win the election if the Hare system is used?
Which column of voters would be most disappointed in the result?
Devise an alternative ranking that they could submit to get a result more to their liking when the Hare system is used?

Voting – Activity Sheet 3 NAME ______________________________

DATE:______________

Tournament Digraphs and Condorcet Winners

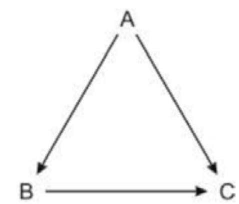
We can use a diagram called a tournament digraph to illustrate the expected results of head-to-head polls among the candidates. We use vertices (dots) to represent each of the candidates and draw an arrow from one candidate to another if the first candidate would beat the second in a head-to-head competition. If no ties occur, exactly one arrow would join every pair of candidates.



Example: The tournament digraph corresponding to the Adventure Club example on the first sheet is sketched to the right.

Note that in the head-to-head matches, skiing beats rafting, rafting beats caving and caving beats skiing.

In the example above, no activity defeated all the other activities in head-to-head voting. Suppose instead that the tournament digraph for a set of preferences looked like this:



1. Who would you expect to win using plurality?__________

   Hare system? __________
   Borda count? ___________
   Sequential pairwise voting? ________

2. Sketch the tournament digraph for the election on the Strategic Voting page. Does this result agree with your response in question 1 on this page?

3. How many arrows are there in a digraph with five candidates?

   With ten candidates?

   With n candidates?

4. Sketch the tournament digraph of the following:

NUMBER OF VOTERS				
RANKING	**38%**	**29%**	**24%**	**10%**
FIRST CHOICE	W	Z	X	X
SECOND CHOICE	X	Y	Z	W
THIRD CHOICE	Y	W	Y	Y
FOURTH CHOICE	Z	X	W	Z

5. Determine the winner of the election in problem 4 above, when the Borda count is used. Is it what you expected?

Note that with four candidates, each first-place vote will be worth 3 points, each second-place vote will be worth 2 points and each third-place vote is one point.

When a candidate beats every other candidate in head-to-head contests, we call the candidate a Condorcet winner.

6. Develop a table of preference lists for which the Condorcet winner would lose the election under the Hare system of voting.