What Goes in the Middle? A Curriculum Unit on Axioms, Conjectures and Logical Thinking

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Overview

This unit is planned to be taught at the beginning of eighth grade algebra. The goal of its activities is to lay a foundation for understanding proof in mathematics. The activities will help students understand what axioms and theorems are, how conjectures are posed, and how logical thinking answers questions. The unit could also be taught to willing sixth or seventh graders or high school algebra students. It would be helpful if the instructor using this unit has written proofs or studied logic as part of his or her own mathematical training, but the unit could still be used if this is not the case.

Rationale

Aristotle said, “All questions are a search for a middle” (McKeon, p. 75). The “middle” is a connected series of statements that connect question to answer and proceed from pre-existing knowledge to the creation of new basic truth. This curriculum unit is designed to set eighth grade algebra students on a search for the “middle” in mathematics.

There has always been a tension between valuing performance and valuing understanding in education. Recently the “math wars” have pitted those who ask, “Can you do it?” against those who ask, “Can you explain it?” If you can do it then you solve a problem by memorizing a rule or algorithm. If you can explain it, then you are able to offer a logical explanation or proof for how you solved it. The push for more reasoning and proof in math curricula actually began in the early years of the 20th century (Herbst, p. 308) and accelerated with the launch of Sputnik and the space race in the late 1950s and early 1960s. Being able to prove a mathematical statement was seen as real competence in mathematics. The anxiety of our Cold War competition with the Soviets led to math and science education reform. Several universities competed to develop mathematics curricula that taught formal logical proof in secondary school. In a 1963 report on these new mathematics programs the National Council of Teachers of Mathematics (NCTM) asked, "At what level should proof be introduced and with what degree of rigor? How rapidly should a student be led to make proofs independently?" (NCTM, 1963, p. 3)

The success of the Apollo missions allowed the focus on "new math" to subside, but in the 1980s and 1990s globalization of the world economy once again focused attention on mathematics and science education. This time proficiency in math for our students was seen as a way to ensure the nation's economic competitiveness (National Research Council, p. 1):

> *"Mathematics is the key to opportunity. No longer just the language of science, mathematics now contributes in direct and fundamental ways to business, finance, health, and defense…To participate fully in the world of the future, America must tap the power of mathematics."*

The push that started with the Carnegie Corporation's "A Nation at Risk" in 1983 (National Research Council, p. 2) led the NCTM to ultimately develop a wide-ranging program of reform. The ability to reason, prove and solve problems is now a fundamental skill in the NCTM's resulting standards.

The National Council of Teachers of Mathematics Curriculum Standards for Reasoning and Proof (NCTM, 2004) says that:

> *Instructional programs from prekindergarten through grade 12 should enable all students to—*
> * *Recognize reasoning and proof as fundamental aspects of mathematics;*
> * *Make and investigate mathematical conjectures;*
> * *Develop and evaluate mathematical arguments and proofs;*
> * *Select and use various types of reasoning and methods of proof.*

It is hard for middle school students to understand the power and precision of the if-then relationship in mathematics. They are emerging from the world of arithmetic and operations and are now being asked to lift their heads and take a wider view. They are not used to making abstract generalizations or making a series of deductions from a set of observations or assumptions. However, they are at a receptive point in their development where they could develop a taste for logical argument. The logical structure of philosophy is similar to that of mathematics, and in the last 30 years researchers here and in Great Britain have developed curricula to teach philosophy to young children and adolescents. They have found that "the study of philosophy can help students to develop their thinking in order to make it both more critical and more creative. Students are challenged to question assumptions, analyze concepts, look at relations between claims and conclusions, evaluate arguments, consider analogies, offer examples, raise counterexamples, and consider the further implications of claims and arguments" (Goering, p. 48).

Middle school teachers are preparing their students to take the challenging math courses available to them in high school as well as laying the groundwork for college entrance exams like the SATs. Knowledge of proof and logical argument can build a foundation for future academic success.

The structure of mathematical proof was assembled in classical times by Greek philosophers and geometers. Aristotle (384-322 B.C.E.) showed how to create a logical argument, and the techniques that he developed in his "Posterior Analytics" are still used. Two of the most basic connectors in the sequence of a proof are *modus ponens* and *syllogism*. *Modus ponens* deals with the if-then statement. Given that the statement "If p then q" is true, then, when p is true, q must be true. More simply we say that p implies q. *Syllogism* joins a series of if-then statements: Given that "if p then q" and "if q then r" are true, then if p is true, r must also be true. We see this defined in the basic rules for equations as the Transitive Property: If a = b and b = c, then a = c.

The Greek mathematician/geometer Euclid flourished in the time around 300 BCE. He is perhaps the most famous "prover" of all time. His book, <u>The Elements</u>, remains fresh and accessible even today because it has been in use continuously since he wrote it for students at his school in Alexandria. The thirteen books of <u>The Elements</u> developed proofs for most of the concepts in geometry. It is arguably the most influential book ever written not only because it established the axiomatic foundations of geometry, but also because it gave humanity a rational way to analyze and solve problems of all kinds.

Burnett Meyer has defined mathematics as the study of axiomatic systems. An axiom is a basic definition, a building block from which other concepts can be built. The statements that follow from axioms are theorems. "Any statement deduced logically from the axioms is called a theorem. A collection of primitive terms, axioms, and all theorems deducible from them is known as an axiomatic system or mathematical discipline. Mathematics is the study of all such axiomatic systems" (Meyer, p. 1).

An example of an axiom in arithmetic is adding zero. Adding zero to a number does not change that number, and in the real number system zero is called the additive identity. In creating proofs in algebra, this basic axiom becomes a stepping-stone from one equation to another. One of the first methods that students learn in solving simple linear equations is to subtract the constant from both sides of the equation. On one side of the equation adding and subtracting a number becomes adding zero. This transformation leaves the term with the variable "alone" and available for the next step in the solution.

Finding ways to reduce part of an algebraic expression to zero thus becomes a powerful tool in proving theorems in algebra.

An analogy for axioms that middle school students could relate to is the process of rule making for the games that they play outside in their neighborhood or schoolyard with their friends. In the 1960s British folklorists Iona and Peter Opie studied how children play games (Opie, p. v) and they were able to list rules for hundreds of games. They collected data from across the British Isles, and they found that the types of games and the rules that children followed were remarkably similar in disparate locations. Fieldwork by other researchers showed similarity in the rules for games played in the United States to those played long ago in Europe. Some games have ancient origins. "Blind Man's Buff" was played in the 14th century (Opie, p. 117). A rough-and-tumble form of leapfrog sometimes called "Buck, Buck" was written about in Roman times (Opie, p. 299), yet it was played by this author's husband on the corner of 53rd Street and Fifth Avenue in the early 1950s in Brooklyn, New York. This robust cultural inheritance is the basis for the first activity of this curriculum unit for two reasons. First, it creates an emotional connection between students and mathematics by linking math to an activity dear to their hearts. Secondly, the rule-making process of games is not unlike that which establishes the axioms of a mathematical system:

> *"Play is unrestricted, games have rules…[Children] like games which move in stages, in which each stage, the choosing of leaders, the picking-up of sides, the determining of which shall start, are almost games in themselves. In fact children's games often seem laborious to adults who, if invited to join in, may find themselves becoming impatient, and wanting to speed them up. Adults do not always see, when subjected to lengthy preliminaries, that many of the games are more akin to ceremonies than competitions." (Opie, p. 2)*

For example, when children have a difference of opinion during a game and arbitrate a new rule that clarifies one or more of the original rules, it is similar to the process that mathematicians use when they recognize theorems that follow from axioms. The logical thinking needed to understand rules and their consequences is a natural human trait that if activated could support children's learning.

Mathematicians prove theorems for a living. For example, in order to get a doctorate, graduate students in math must have a good problem to solve, and often one is suggested to them by their thesis advisor. Other times, a mathematician notices a pattern in some results and comes up with a possible explanation, similar to the way a physical scientist observes a pattern in data that has been collected and has an idea of why it might be true. The scientist sets up an experiment to prove or disprove his or her hypothesis. Likewise, the mathematician poses a conjecture and then tries to prove it true or false using their knowledge of mathematics and logic. Sometimes, the ability to prove a conjecture is beyond even the person who first posed the problem, and the problem waits for others to solve it. Sometime in 1637 (Singh, p. 62) the French mathematician Pierre de Fermat wrote in the margins of a book a conjecture about the solutions to the equation

$$x^n + y^n = z^n$$

He said he had a proof, but it was too lengthy to write there in the margin. Time passed, and he neglected to explain his proof later. The proof died with him in 1665. Over the course of the next 300 years mathematicians tried to figure out what Fermat's reasoning was. It was not until 1995 that British mathematician Andrew Wiles published the first accepted proof of Fermat's conjecture (Singh, p. 279).

Another conjecture that took a long time to prove is from the field of topology. In map-making, cartographers distinguish the border of one country from one another by coloring adjoining lands in different colors. It had been noticed that mapmakers could accomplish this goal with a minimum of four colors. This was posed as a mathematical conjecture in 1852 (Devlin, p. 188), but not solved until 1976 when computers made possible the thousands of calculations required.

Proof in mathematics is based on making accurate logical arguments. Since ancient times puzzles and brainteasers have taught logical thinking in an entertaining way. Martin Gardner says of puzzles, "…the logic that solves them is essentially like the reasoning that mathematicians and scientists use when confronted with a perplexing question…the solving of amusing logic puzzles is good training for solving more serious problems" (Gardner, aha! Insight, p 89).

One form that these puzzles often take is truth-teller vs. liar. For example:

> A sailor is shipwrecked on the shores of a faraway country. There, all the inhabitants belong to one of two clans, Truth-tellers or Liars, but an observer couldn't tell by someone's appearance alone to which clan they belong. Truth-tellers must always tell the truth and Liars must always lie. The sailor needs to get to town to seek help and comes to a fork in the road. A local inhabitant passes, and the sailor, being a clever person who has sailed the world many times, has heard about the local customs in regard to honesty. In asking for directions, he asks the person but one question. The local's answer was, "Take the left turn." The sailor then waved goodbye and set off confidently to the right. What question did the sailor ask?

The solution is that the ingenious sailor phrased the question in this way: "How would a member of the *other* clan answer if I were to ask which road to take?" The town is actually to the right, so if the inhabitant were a Truth-teller, he or she must say that a member of the other clan (the Liars) would say the left road. On the other hand, had the sailor encountered a member of the Liar clan he would also have been told to go to the left. A Truth-teller would say to take the right turn and the Liar must turn this advice on its head. Therefore the sailor knows that whatever he is told, he must take the *opposite* road.

Drawing a table or diagram that shows all possible outcomes can be helpful in solving these puzzles. For example:

Clan	Real Direction	*Other* clan says to go	Sailor goes
Truth-teller	Right	Left	Right
Liar	Right	Left	Right
Truth-teller	Left	Right	Left
Liar	Left	Right	Left

Many variations on this problem have been published in books and on the Internet. They are sometimes posed as "Knights" (who tell the truth) and "Knaves" (who lie) puzzles. Sources for additional puzzles to solve are listed in the bibliography.

Objectives

The first objective of this curriculum unit to give students an understanding of what is meant by "axiom" and "theorem." The second is to show how mathematicians come up with theorems that they would like to prove, how conjectures are developed. Third is to give students experience with solving problems using logic, "if-this-then-that" type of thinking. Fourth, this curriculum unit will give students opportunities to make oral presentations in support of their ideas and to make their logical connections clear to others. Finally, the unit will teach some organization skills that mathematical researchers use in real life to be efficient and accurate in their work.

Strategies

For the Teacher

It is beyond the scope of this curriculum unit to provide comprehensive instruction on how to construct a proof or give a detailed history of major mathematical proofs. Thus, an important strategy for the instructor would be to spend some amount of preparation time reviewing the basics of professional mathematics proof and solving some classic mathematics problems, perhaps working with a colleague. The annotated bibliography gives several suggestions for books and websites to consult. Some of the books are classics, like George Polya's How to Solve It. In it, Polya explains the major approaches to solving problems and doing proofs.

Some high school textbooks are great sources of problems and have problem solving suggestions for students as well as chapters on doing proofs. The annotated bibliography lists a few.

There are many websites to visit, but "Making Mathematics: Mentored Research Projects for Young Mathematicians" by the Educational Development Corporation is exceptional and highly recommended. From 1999 to 2002, under a grant from the National Science Foundation, EDC's project "matched students and teachers in grades seven through twelve with professional mathematicians who mentored their work on open-ended mathematics research projects." Their excellent teacher's manual defines proof and provides detailed historical and mathematical background. It shows problems that students worked on, and provides solutions as well as samples of student work. It shows how to put into practice important pedagogical approaches like keeping mathematical research notebooks and peer review of work.

Other recommended websites are the Math Forum, Cut-the-Knot and Ivars Petersen's Math Trek online. See the annotated list of resources for more information. Also recommended are the websites for middle school and high school math competitions. These are sources for sample problems and your implementation of this curriculum unit may encourage students to enter challenging math contests.

The Student Notebook

In order to foster organized thinking and writing, an important strategy is to teach students to keep a good notebook. In the course of doing their work, mathematicians and scientists need to keep records of their daily output. For some projects, the standards for maintaining notebooks are quite stringent, requiring signatures of witnesses to data and pasting written work from other sources into the official notebook's pages. It is possible that disputes could come up later, and if a person can prove when or how they came to a result, the records in their notebooks could resolve a patent challenge or other academic or legal questions. Mathematicians are used to pursuing a line of thought that they abandon because it brings them to a dead-end. Later, however, they may want to go back and revisit a previous approach. They are able to do so if they have kept good notes of their work. Both mathematicians and scientists hope to eventually publish their work in a respected journal, and their notebooks are their sources for their writing.

So, during this curriculum unit students will be expected to maintain a daily notebook. They should record notes from teacher instruction and class discussions, outlines and rough drafts of written assignments, "Notes to Self" with insights and ideas to research or pursue, work on solutions to problems, definitions for mathematical terms, diagrams or drawings used in solving problems, citations for any books or websites that they consult and any other useful information they may encounter. On the first day of the unit, guidelines for using the notebook should be copied and distributed to students at the (see Appendix A for the Notebook Instructions).

Content Strategies

The concepts in this unit will be taught in three segments. First is a written project that appeals to what students already know about axioms and theorems by exploring the rule-making and arbitration processes of games that they have played with their friends. Second, students will explore the Four-Color Map Theorem in order to understand how a conjecture is posed. Third, solving logic puzzles will provide experience in making structured arguments.

Oral Communications

In each activity students will either join in guided class discussion analyzing the activity or they will make oral presentations to their classmates of their results. This is important for many reasons. Students learn from each other, and explaining thoughts out loud about abstract ideas helps to make those ideas more comprehensible. Making oral presentations – the art of public speaking – is an important skill for future mathematicians and scientists.

Classroom Activities

Day One -- Introduce Rules of the Game Project

The unit is introduced by a writing project that asks the students to define the rules and consequences of a favorite neighborhood childhood game. The purpose of this activity is to make students realize that they have already been using axiomatic systems that employ rules or axioms, definitions, and theorems in a fashion similar to mathematics. The types of games that could be used for this project are endless – any variety of tag, hide-and-seek, hopscotch, jump-rope, keep away, playing "house" or "school," capture-the-flag and the like.

Begin the lesson with a class discussion about playing games. Have ready a sheet of poster paper to list students' answers to questions. Start with some basic questions:

- What is a game?
- What are some games that you play in your neighborhood?

The discussion is bound to be lively. Let the students know that although there are many formal games with well defined rules, like basketball and football, board games like Monopoly and games played with a deck of 52 cards, what this project is about is a game that is played outdoors on the sidewalk, street or backyard. After some discussion, distribute the project assignment to the class. Go over the requirements for the assignment. This is a homework project that should be completed in about a week to ten days. While students are working on this assignment at home, work will continue in class with the other aspects of the curriculum.

Rules of the Game Project Handout (Copy for students)

For this project, think back to a time when you were younger, or maybe last summer when you played games in your neighborhood with your friends or family. Not a formal game like baseball or basketball and not a board game like Monopoly or Scrabble, but a game that kids teach to other kids. Choose one game to be the topic of this project. Think about how you played it and what the rules were. It doesn't have to be a complicated game. The important thing is to be very detailed in your explanation. Follow the directions below. Assemble the rules and diagrams of the game neatly on a poster that you can refer to when you present your project to the class. Accompany your poster with a written report that includes the following information:

I. Introduction

In a few paragraphs give the reader an overview of your game. What is the name of the game? How did you learn this game? Who taught you to play? Where could you play?

II The Rules

Write the rules for your game. Be as detailed as possible. Consider the following questions to help you define the rules, but do not feel limited by them.

- Describe the players: Who could play? How many could play?
- How were the players gathered? How did they know a game was about to be played?
- How did you decide who went first? What rhymes, if any, did you say to choose who would be "it" or go first?
- How did the game start?
- How was the game played?
- Were there any geographic boundaries?
- Were there any objects that were used in the game?
- Were there any exceptions to the rules?
- Were there any rules to cover special circumstances?
- How do you know who won?
- When was the game over?

III Make two or three drawings or diagrams to help explain your game.

IV Resolving Disputes

Describe how disputes were resolved. If players disagreed, was there a formal way of making a decision about who was right, or did someone act as a referee? Think of a time when there was an argument and explain how it was decided. Did one rule take precedence over another?

III. Conclusion
In a short paragraph, summarize your feelings about the game. What did you like about this game – why was it fun? Did you ever make up new rules or variations for the game? Were there any other games with similar rules?

Day Two -- The Four-Color Map Theorem

The second activity of the curriculum unit is an exploration of the Four-Color Map Theorem. The purpose of this activity is to teach students what it means to make a conjecture. How do mathematicians come up with a question that they would like to investigate? How do we define a conjecture that needs to be proved?

The Four-Color Map Theorem says that a plane surface that is divided into contiguous areas can be colored with at most 4 different colors. Each adjacent area must be colored with a different color, but a joining at a single point is not considered to be adjacent. The theorem was only recently proved by "brute force," using a computer to examine all possible ways of assembling a map and showing that four colors were all that were necessary to color it.

For the activity, supply students with a blank map of the United States. You can ask one of your social studies colleagues for a blank map, or find one available free on the Internet. Houghton Mifflin Harcourt's Education Place has a nice US map at http://www.eduplace.com/ss/maps/usa.html.

Ask each student to choose four colored pencils. Challenge them to color the map with only these four colors. They should approach the problem somewhat systematically, since jumping around on the map may incorrectly color two adjacent regions the same.

Once students have completed their maps, hang them up around the room and give them time to examine each other's work. Then ask students to respond to the assignment in writing. Their written response in their notebooks should try to answer questions like: What is a map? What is a boundary? What does it mean to be adjacent? Would three colors have been sufficient to color the map? If not, where on your map was it necessary to use four colors? Why do you think this is a mathematical question?

Day Three – Four-Color Map Follow-up

This activity needs to take place in a library or computer lab with Internet access. In the bibliography are references to websites about the Four-Color Problem that students should read. Then they need to finish the written response to the assignment by answering these prompts:

- Explain clearly why the region on your map that you identified yesterday needed more than three colors.

- Experiment. Try to draw a part of a map that would need five colors. Is it possible? Why or why not?

The purpose of this assignment is not so much to try to prove this conjecture, but to understand how the problem is defined.

Day Four – Logic Puzzles

This classroom activity will ask students to solve a classic genre of logic puzzles -- one that sets truth-tellers against liars. The solver usually has to identify who is telling the truth and who is lying. This activity begins with students solving several variations on this puzzle and then asks them to create their own version.

Ask students to copy the puzzle described earlier in the Rationale section into their notebooks:

> A sailor is shipwrecked on the shores of a faraway country. There, all the inhabitants belong to one of two clans, Truth-tellers or Liars, but an observer couldn't tell by someone's appearance alone to which clan they belong. Truth-tellers must always tell the truth and Liars must always lie. The sailor needs to get to town to seek help and comes to a fork in the road. A local inhabitant passes, and the sailor, being a clever person who has sailed the world many times, knows about the local customs in regard to honesty. In asking for directions, he asks the person but one question. The local's answer was, "Take the left turn." The sailor then waved goodbye and set off confidently to the right. What question did the sailor ask?

Assign them to work alone on it for a set period of time, say 10 minutes. Then break them into groups to come to a final solution. When students ask for help, don't offer specific advice on the solution, but suggest strategies they could use: make a drawing or table or chart, act it out, try a tree diagram, etc. Ask each group to prepare an overhead of their solution. Have each group present their solution. Ask them to emphasize the phrase "if-then" in their presentation.

When the presentations are over, discuss and summarize the problem-solving strategies that students used.

Day Five – More Truth-teller vs. Liar Problems

Using the resources that are suggested in the bibliography, identify two or three additional Truth-teller vs. Liar problems to solve, like the following:

On Freshman Day, each new student had the option of being a Truthteller or a Liar for the day. The Truthtellers had to speak only the truth; the Liars would speak only lies. I came upon three freshmen, A, B and C, sitting on a step. I asked A whether he was a Truthteller or a Liar. A answered with his back turned, so I could not hear what he said. "What did he say?" I asked B. B said, "A says he is a Truthteller." C said, "B is lying." Was C a Truthteller or a Liar? (Averbach and Chein, pp. 38-39)

[Solution: C is a Liar]

John and Bill are residents of the island of Knights and Knaves.

Question 1
John says, "We are both knaves."
Who is what?

Question 2
John says, "If (and only if) Bill is a knave, then I am a knave.
Bill says, "We are of different kinds."
Who is what? (http://en.wikipedia.org/Knights_and_Knaves, p. 2)

[Solution to Question 1: John is a knave and Bill is a Knight. Solution to Question 2: John is a Knave and Bill is a Knight.]

Ask students again to write the problems into their notebooks and write their solutions. Discuss their solutions. Pose questions for the class: "Who can tell us how they proved that C is a Liar? Who can tell us how they proved that John is a Knave and Bill is a Knight? Asking the questions in this way gets around the issue of being right or wrong and moves on to showing how to justify the solution. Students should be careful to take notes on this discussion since it may help them to write their own problem

Finally, students should create their own Truth-teller vs. Liar problem. This should be assigned as homework. The problem and its solution should be written on separate pieces of paper to be used in the next class.

Day Six – Students' Truth-teller vs. Liar Problems

Divide the class into small groups and ask them to exchange problems. Each student should try to solve another's problem. They should be writing the problem and their solutions in their notebooks. When they are done, have the authors consult with the solvers to check their solutions. Then come together as a class to discuss the project.

As time permits, ask each group to share a problem that they thought was especially creative or tricky.

Day Seven – Presentation of Rules of Games Project

Using their posters as a visual, have each student present his or her game to the class. This is a suggested protocol for presentations:

- Listen politely and attentively to the presenter.
- When he or she is finished, you may ask questions.
- Questions can be followed with positive comments about the project.
- Applaud when the presenter is done.

It may take more than one class period to give all students a chance to make their presentation.

After all the students have presented their games projects, the class as a whole should have a discussion. Some possible prompts for the discussion: What did your games have in common? How were the games different? How important was it to follow the rules? Do kids in other neighborhoods play the same game? Do they follow the same rules?

Day Eight – Rules of the Game Project Wrap-Up

Finally, and most important, ask the students to compare the rules of their children's games to the rules for math. This activity should not be rushed. As a group, ask the students to define some basic rules for simple operations like addition and subtraction. Write these on a poster so that they can be saved. It is not necessary to be comprehensive. Then ask the students to brainstorm similarities and differences: How are the rules for games and math the same? How are they different? Again, as a whole class, have the students share their ideas and write them on a poster. Ask the students to decide which rules are axioms, the most basic. What rules are theorems that are dependent on axioms for their meaning?

Annotated Bibliography/Resources

Averbach, Bonnie and Chain, Orin. Mathematics, Problem Solving through Recreational Mathematics. New York. W. H. Freeman and Company. 1980. A comprehensive source of problems, arranged by category, with sample solutions and answers.

Bergmann, Merrie, Moor, James and Nelson, Jack. The Logic Book. New York. McGraw-Hill Publishing Company. 1990. A good reference on symbolic logic.

Bogomolny, A. Manifesto. Interactive Mathematics Miscellany and Puzzles from Interactive Mathematics Miscellany and Puzzles http://www.cut-the-knot.org/manifesto/index.shtml. Accessed 14 June 2009. A problem solving website, dedicated to the notion that "the ability to appreciate Mathematics enhances the lives of those who possess it."

Devlin, Keith. Mathematics: The Science of Patterns. New York. Scientific American Library, A division of HPHLP. 2004. This book tries to convey "the essence of mathematics, both its historical development and its current breadth." It has a clear discussion of the four-color map theorem.

Drexel University. The Math Forum. http://www.mathforum.org. Drexel University. 2009. The Math Forum is a research and educational enterprise of the Goodwin College of Professional Studies. A comprehensive mathematics resource with a focus on problem solving. Sign up for the excellent digital newsletter.

Educational Development Corporation. Making Mathematics, Mentored Research Projects for Young Mathematicians. http://www2.edc.org/makingmath. 2003. The website for an amazing project supported by a grant from the National Science Foundation. The teacher handbook details a comprehensive pedagogy of proof. A must-read.

Gardner, Martin. aha! Insight. New York. Scientific American Inc. W. H. Freeman and Company. 1978. Classic problems, arranged by category with solutions and explanations.

Gardner, Martin. Richards, Dana, ed. The Colossal Book of Short Puzzles and Problems. W. W. Norton and Company. New York. 2006. A great compendium of problems and solutions by the Scientific American author. It includes a discussion of truth-tellers vs. liars problems.

Gardner, Martin. New Mathematical Diversions from Scientific American. New York. Simon and Schuster. 1966. More engaging problems. Chapter Four describes Lewis Carroll's contributions to the truth-teller genre. Chapter Ten discusses the history and mathematics of the four-color map theorem – written before the theorem was proved.

Goering, Sara. "Finding and Fostering the Philosophical Impulse in Young People: A Tribute to the Work of Gareth B. Matthews." Metaphilosophy. Vol. 39, No. 1, January 2008.

Heath, Sir Thomas L. tr. The Thirteen Books of Euclid's Elements, 2nd edition. Dover Publications, Inc. New York. 1956. If you have never actually done so, read at least Book 1 of The Elements. The commentary of Sir Thomas L. Heath is a rich treasury of the history of mathematics and geometry.

Herbst, Patricio G. "Establishing a Custom of Proving in American School Geometry: Evolution of the Two-Column Proof in the Early Twentieth Century." Educational Studies in Mathematics. Vol. 49. No. 3. 2002

Herr, Ted. Johnson, Ken. Problem Solving Strategies, Crossing the River with Dogs. Key Curriculum Press. Emeryville, California. 1994. The textbook for a course on problem solving methods and a wonderful source of problems. The chapter on matrix logic is applicable to truth-teller vs. liar problems.

Math Counts Foundation. https://mathcounts.org/Page.aspx?pid=195 Website for the Math Counts Foundation which sponsors the Math Counts competition.

Mathematical Association of America. MAA Online. http://www.maa.org/news/columns.html. A huge archive of articles devoted to math topics.

McKeon, Richard. Introduction to Aristotle. The Modern Library. New York, New York. 1947.

Mathematical Association of America. MAA Online. Competitions. http://www.maa.org/subpage_6.html Website devoted to a range of mathematics competitions. The middle school competition is the AMC8.

Meyer, Burnett. An Introduction to Axiomatic Systems. Prindle, Weber & Schmidt, Incorporated. Boston, Massachusetts. 1974. An elegant overview of the foundations of mathematics.

National Council of Teachers of Mathematics. An Analysis of New Mathematics Programs. Washington, D.C. 1963. A report on eight "new math" curricula.

National Council of Teachers of Mathematics. Principles and Standards for School Mathematics. 2004. http://standards.nctm.org/document. The math teacher's bible.

National Research Council. Everybody Counts, a Report to the Nation on the Future of Mathematics Education. National Academy Press. Washington, D.C. 1989. The seminal study.

Opie, Iona and Peter. Children's Games in Street and Playground. Oxford University Press. London. 1969. A marvelous study of children's games.

Peterson, Ivars. "Coloring Penrose Tiles." Science News Online. May 15, 1999. http://www.sciencenews.org/sn_arc99/5_15_99/mathland.htm
Another website for the four-color map assignment.

Peterson, Ivars. "Maps of Many Colors." Science News Online. 1996. http://www.sciencenews.org/pages/sn_arc97/1_4_97/mathland.htm The primary website for the four-color map assignment.

Polster, Burkhard. Q.E.D. Beauty in Mathematical Proof. New York. Walker & Company. 2004. A beautiful little book about proof.

Polya, G. How to Solve It, A New Aspect of Mathematical Method. Princeton University Press. Princeton, New Jersey. 1985. Polya is the master teacher of problem solving.

Schwartz, Diane Driscoll. Conjecture & Proof, An Introduction to Mathematical Thinking. Fort Worth. Harcourt Brace College Publishers. 1997. A college text.

Serra, Michael. Discovering Geometry, An Inductive Approach. Key Curriculum Press. Berkeley, California. 1997. A great high school geometry textbook. A clear discussion of Euclid and using logic in proof.

Shasha, Dennis. The Puzzling Adventures of Doctor Ecco. Mineola, New York. Dover Publications, Inc. 1998. Chapter Two has some truth/lie problems.

Singh, Simon. Fermat's Enigma. New York. Walker and Company. 1997. Sub-titled "The Epic Quest to Solve the World's Greatest Mathematical Problem" this book is an engaging history of the 300 year search for a proof to "Fermat's Last Theorem."

Smith, Douglas, Eggen, Maurice, and St. Andre, Richard. A Transition to Advanced Mathematics. Pacific Grove, California. Brooks/Cole Publishing Company. 1990. A college text on proof.

Solow, Daniel. How to Read and Do Proofs. New York. John Wiley and Sons. 1990. Another college text.

Stevenson, Frederick W. Exploratory Problems in Mathematics. Reston, Virginia. National Council of Teachers of Mathematics. 1992. Some excellent problems to solve.

Stewart, Ian. How to Cut a Cake and Other Mathematical Conundrums. New York. Oxford University Press, Inc. 2006. Chapter Four discusses truth-teller/liar problems and there are several references to the four-color map problem.

Thomas, Robin. The Four Color Theorem. http://people.math.gatech.edu/~thomas/FC/fourcolor.html A faculty member at Georgia Tech University's website includes this article about the Four Color Map Theorem. Although some parts are advanced, the beginning is accessible for the assignment.

Annotated Reading List for Students

Blum, Raymond. Mathamazing. Sterling Publishing Co., Inc. New York. 1994. A fun collection of puzzles and riddles.

Burns, Marilyn. The I Hate Mathematics! Book. Little, Brown and Company. Boston. 1975. Lots of problems with good explanations and amusing illustrations.

Burns, Marilyn. Math for Smarty Pants. Little Brown and Company. Boston. 1982. More problems with suggestions for how to solve them. Fun illustrations.

Gardner, Martin. My Best Mathematical and Logic Puzzles. Dover Publications, Inc. New York. 1994. Another collection of puzzles and problems collected by a master puzzle poser.

Suggested Materials for Classroom Use

Student notebooks
Write-on overhead slides and projector
Blank maps of United States
Colored pencils
Poster paper

Appendix A

Notebook Instructions (Handout)

1. For this curriculum unit you will need to purchase a bound notebook with graph paper pages. This is sometimes referred to as "quadrille" paper. These notebooks are available at most office supply stores and resemble a marble composition book. The graph paper format will help to keep your writing organized, particularly if you make tables or charts.
2. When you first get your notebook, write your name in permanent marker in the space provided on the cover. Inside the notebook write only with pen. Number the front of each page sequentially in the upper right hand corner.
3. Leave three pages blank at the beginning to be used as a Table of Contents.
4. Use the back of the book for your glossary, starting with the last page and working your way backward. Write down any word that needs to be defined along with its definition.
5. Each day, start a new page and write the date in the margin on the left.
6. If in the course of your work you make a mistake, draw a line through it and write the correction next to it. Later you may find it wasn't a mistake after all, and you may want to read what you wrote previously.

7. What should you write in your notebook? The following is a list of suggestions:
- Notes from your teacher's lessons
- Instructions for activities or assignments
- Notes from class discussions
- Outlines and rough drafts of written assignments
- "Notes to Self" with insights and ideas to research or pursue
- Work on solutions to problems
- Definitions for mathematical terms
- Diagrams or drawings used in solving problems
- Citations for any books or websites you consult in the course of your work
- Any other useful information you would like to save

8. Work on specific problems should be recorded in this way:
- Give the problem a name and write it down as a heading.
- Write the source of the problem (teacher handout, book title with page #, website URL, etc.).
- Copy the problem word for word into your notebook, along with specific questions that need to be answered.
- Show all work for the problem. In the event that calculations were made, write down the expressions that you entered into the calculator to get your answer.
- Don't be afraid if it seems a bit messy or disorganized.
- Make tables, charts, or drawings to help visualize elements of the problem.
- Write the answers to questions clearly in complete sentences.
- Assignments that need to be handed in will be copied over neatly or typed on the computer from the work in your notebook.

9. Keep your Table of Contents current by writing dates on the left, names of topics or problems in the center, and page numbers on the right.

Content Standards

The applicable content standards from the Academic Standards for Mathematics published by the Pennsylvania Department of Education in June 2008 are:

2.4. Mathematical Reasoning and Connections

2.4.8. GRADE 8

A. Reasoning: Draw **inductive** and **deductive** conclusions within mathematical contexts.

B. Connections: Use if…then statements to construct simple **valid arguments**.

2.5. Mathematical Problem Solving and Communication

2.5.8. GRADE 8

A. Problem Solving: Develop a plan to analyze a problem, identify the information needed to solve the problem, carry out the plan, apply estimation skills as appropriate, check whether the plan makes sense and explain how the problem was solved in grade appropriate contexts.

B. Communication: Use precise mathematical language, notation and representations, including numerical tables and **equations**, simple algebraic **equations** and formulas, charts, graphs and diagrams to explain and interpret results.