

ALGEBRAIC EQUATIONS

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OVERVIEW:

The purpose of this project is to familiarize students with the idea of “Algebraic Equations” and their solutions. Algebraic Equations project is geared towards grade level 9th – 12th. This project contains the definition of an equation and concept of variables in length. Classification of equations is done on different basis like degree of equations, number of variables and reducible equations. Solving of linear equations and quadratic equations is explained by different methods such as;

- Substitution Method
- Elimination Method
- Graphical Method
- Discriminant Method

RATIONALE:

Equations play a very dominant role in the study of any branch of science, arts, environment, economy or educational puzzles. By the word equation, literally speaking, we mean the equality of two statements. For example, when a number is added to the $\sqrt{121}$; it becomes equal to 20. The unknown number is assigned a letter symbol “x” and problem is given a mathematical model and solved for the value of variable “x”. Equations always involve variables. Equations involving only one variable are called equations in one variable, like wise in two variables x and y: three variables x, y and z. In solving simultaneous equations usually we need as many numbers of equations as we have the variables.

For Example,

- $x + y = 5$ will have many solutions but if the same equation is solved simultaneously with $x + 3y = 7$, we need two equations and system can be solved. $x = 4$ and $y = 1$ is the only unique solution.

Substitution Method: The substitution method is used to eliminate one of the variables by replacement when solving a system of equations. Think of it as "grabbing" what one variable equals from one equation and "plugging into the other equation.

For Example,

- $x + y = 5$ and $x + 3y = 7$

Using this method, one variable is expressed in terms of another:

$$y = (5 - x)$$

Plugging in the value of y in other equation

$$x + 3(5 - x) = 7$$

Solving for x, we get $x = 4$

Plugging in the value of x in $y = 5 - x$ gives $y = 1$. Therefore, (4, 1) is the solution.

Elimination Method: In this method, coefficients of one of the variables are made equal by multiplying with suitable numbers to the two equations and by adding or subtracting one of the variables is eliminated.

For Example,

- $2x + 3y = 10$

$$5x - 2y = 6$$

In the above system of equations, to make the coefficients of y equal, we multiply the top equation by 2 and bottom equation by 3 and add:

$$4x + 6y = 20$$

$$15x - 6y = 18$$

$$19x = 38, x = 2$$

Again plugging in the value of x in top equation, we get, $4 + 3y = 10$, $y = 2$. Therefore, (2,2) is the solution.

Another classification of equations is based on the degree of polynomial involved in the equation. We name different degree equations by different names such as;

- $x + 2 = 7$ (Linear, because degree of polynomial involved is 1)
- $x^2 + x - 6 = 0$ (Quadratic, because degree of polynomial involved is 2)
- $x^3 + 3x^2 + 3x + 1 = 0$ (Cubic, because the degree of polynomial involved is 3)
- $x^4 - 1 = 0$ (Biquadratic, because the degree of polynomial involved is 4)

As the degree of equation increases, the complexity in solving or finding the different solutions (or roots) also increases .The most simple type of equations in single variable is linear equation in one variable i.e. $ax + b = 0$, where a is not equal to 0.*This is the general equation in one variable.*

The goal of writing about the linear equations and quadratic equations is to make our learners succeed in this challenging and changing world. To expand the knowledge of the use of linear equations, and how the equations reflect the different ways people work and learn together, students will be prepared both for the real world challenges and their studies in mathematics at college and higher level studies.

The unit on Linear Equation is a part and parcel of “School District of Philadelphia” curriculum for 9th – 12th graders. Unit on linear equations involved in two variables can be explained by different methods fitting into the curriculum for 9th and 10th graders. However, the quadratic equations are explained for higher levels because of the involvement of requisites like; real and imaginary solutions. Less emphasis is drawn upon complex roots of quadratic equations.

Linear equations can be used almost at all levels of studies, depending upon the mathematical modeling of the problems .Linear equations are at the very heart of Algebra. An equation is always treated like a balance with its two pans always balanced about the pivot. One can make changes of any type on both the sides of the sign of equality. Different types of manipulations can be performed on both the sides of an equation: such as,

1. Addition
2. Subtraction
3. Multiplication
4. Division, with the exception of dividing by zero
5. Taking the positive square root with the exception of square root of a negative number

6. Doing the inversion

Example:

1. $x - 3 = 22$; $x - 3 + 3 = 22 + 3$ $x = 25$
2. $x + 4 = 29$; $x + 4 - 4 = 29 - 4$ $x = 25$
3. $\frac{1}{2}x = 3$; $2 \times \frac{1}{2}x = 2 \times 3$ $x = 6$
4. $3x = -12$; $3x / 3 = -12 / 3$ $x = -4$
5. $x^2 = 25$; $\sqrt{x^2} = 5$ or $x = 5$
6. $\frac{1}{x} = \frac{1}{7}$; $x = 7$

Examples (1) - (6) correspond to above mentioned manipulations.

Another type of linear equation systems can be obtained by using two variables or three variables for which we need two simultaneous or three simultaneous equations to solve the systems. For Example: $x + y = 13$; $x - y = 7$ is a linear equation system in two variables x and y which can be solved together to give $x = 10$, $y = 3$. However,

$$x + y + z = 5$$

$2x - y + 3z = 10$ } is a system of linear equations in three variables x , y and z which can be solved together to give $x = 1$, $y = 1$ and $z = 3$.

$$x - y + z = 3$$

It is very important for the students to acknowledge that some problems can be solved without concentrating much on the linear equations. The students may need to define variables and write equations for the simplest of problems to feel comfortable specially when they encounter difficult equations. The following steps are encouraged when one encounters a difficult equation;

- a) Choose the variable you are going to use in each problem and state what it represents.
- b) Write an equation, using your variable that represents the problem.
- c) Solve the equation and the problem using any method you wish, including guess and check (also called trail and error method)

For Example:

1) A chef puts several batches of cubes into a cauldron. The first batch contained 27 cubes and the last batch contained 56 cubes. A total of 108 cubes were put into the cauldron. How many cubes did the chef throw that were not part of first or last batch?

Solution: Let us consider the number of cubes that chef throws and were not the part of first or last batch be “x” (a variable).

Therefore;

$$27 + x + 56 = 108 \text{ (writing an equation)}$$

$$x + 83 = 108 \quad \text{(combining the like terms)}$$

$$x + 83 - 83 = 108 - 83 \text{ (solving for the variable)}$$

$$x = 25 \quad \text{(solution)}$$

Thus, the chef threw 25 cubes that were not the part of first or last batch.

$$\text{Check } 27 + 25 + 56 = 108 \text{ (True)}$$

Therefore, the solution is correct.

Example # 2:

Each adult needed 5 yards of shoelace for the trip to California and each child needed 3 yards. A certain family with seven children needed 71 yards of shoelace. How many adults were in the family?

Solution: Let number of adults in the family be “x” (assigning a variable)

$$\text{Therefore, } 5x + 3 \times 7 = 71 \text{ (Writing an equation)}$$

$$\text{Or, } 5x + 21 = 71$$

$$\text{Or, } 5x + 21 - 21 = 71 - 21$$

$$\text{Or, } 5x = 50 \text{ (Solving the Equation)}$$

$$x = 10$$

Thus, there were 10 adults in the family.

CHECK:

$$5 \times 10 + 3 \times 7 = 50 + 21 = 71 \text{ (True)}$$

Therefore, the solution is correct.

We have seen that a linear equation in one variable such as $3x + 4 = 2(x - 1)$ has a unique solution i.e. $x = -6$. The equation $x + 2y = 5 + 3x + y$ is also a linear equation, but it includes two variables and has more than one solution. The solutions of such an equation can be found easily by assigning one variable in terms of other. For example, the above equation can be simplified to $y = 2x + 5$ and y has been expressed in

terms of x. Assigning different values to variable “x” , we can determine different values for variables “ y ”.

x	0	1	2	-1	-2	-3
y	5	7	9	3	1	-1

However, if these points or ordered pairs (0, 5), (1, 7), (2, 9)are plotted on a graph sheet and joined by a ruler, they make a straight line. A straight line is made up of an infinite number of points, so each point on the line corresponds to solution of this equation. So, we can conclude *a linear equation in two variables has many solutions.*

Review that the term linear has both a “geometric” and an “algebraic” meaning. *Geometrically*, linear means, a straight line and *algebraically*, linear means, the degree of variable used in the equation is one. For example $x + 2y = 5$ is linear but $x + \sqrt{y} = 5$ is non – linear.

Exercise:

A. Classify the following as linear or non – linear equations

- a) $2(x+2) = 3$
- b) $2(x^2 - 1) = 23$
- c) $x - 5 / 3 = 2x - 3 / 4$
- d) $x + 3(x - 2) = 7$
- e) $x^3 + 7x + 8 = x^2 - 3x + 5$

Answers:

- a) Linear
- b) Non Linear
- c) Linear
- d) Linear
- e) Non linear

Exercise:

B: Solve the following linear equations in one variable

- a) $4(y - 3) + 1 = 2(y + 4) - 3$
- b) $3(x + 7) - 5 = 2(x - 5) + 7$
- c) $2x / 3 + 5/6 = x / 6$
- d) $-5(2 - x) = 15$
- e) $4(2x - 3) - 5 = 3 + 5x - 2(x + 1)$

Answers:

- a) $y = 8$
- b) $x = -19$
- c) $x = -5/3$
- d) $x = 5$
- e) $x = 3.6$

QUADRATIC EQUATION

Another class of equations is called “Quadratic equations in one variable”. The most general quadratic equation is $ax^2 + bx + c = 0$; where, a is not equal to 0 and a, b, c are real numbers and “x” is a variable. All the quadratic equations do not find their solutions in the real number system. It depends upon the value of “ $b^2 - 4ac$ ” which is called the Discriminant of the quadratic equation. If $b^2 - 4ac$ is negative, solution of quadratic equation is not present in real number system but is present in complex number system (beyond the scope of this project). However, the nature of solutions or nature of the roots of the quadratic equation can be discussed with the help of discriminant.

1. If discriminant is positive, the solutions or roots are real and unequal
2. If discriminant is negative, the solutions are non-real
3. If discriminant is zero, the solutions are equal

Examples:

1. $2x^2 - x - 6 = 0$; $b^2 - 4ac = 49 > 0$, the roots are real and unequal and by solving we get $x = 2$ and $-3/2$.
2. $x^2 + x + 6 = 0$ $b^2 - 4ac = -23 < 0$, no real roots.
3. $x^2 - 6x + 9 = 0$; $b^2 - 4ac = 0$, the roots are equal i.e. $x = 3$ and 3 .

Solution of a quadratic equation will be explained by different methods in the chapter at a later stage.

OBJECTIVES:

The main objective of this unit is to master the concept of linear equations and quadratic equations.

- A. To explain and make students acquainted with the definition of linear and quadratic equations.
- B. To explain the meaning of solution of an equation
- C. To explain with examples the different methods of solving linear equations and explaining the advantages of using one method over the other :
 - Graphical method of solving two simultaneous linear equations
 - The substitution method
 - The elimination method
- D. To explain the number of solutions of different systems of linear equations
 - Consistent system – unique solution
 - Inconsistent system - no solution
 - Dependent system – many or infinite solutions
- E. List of standards is given in the lesson plans

STRATEGIES:

The above mentioned objectives can be achieved by using different strategies like,

- a) Alternative teaching strategy
- b) Interdisciplinary connection
- c) Inclusion strategy
- d) Enrichment
- e) Co- operative learning

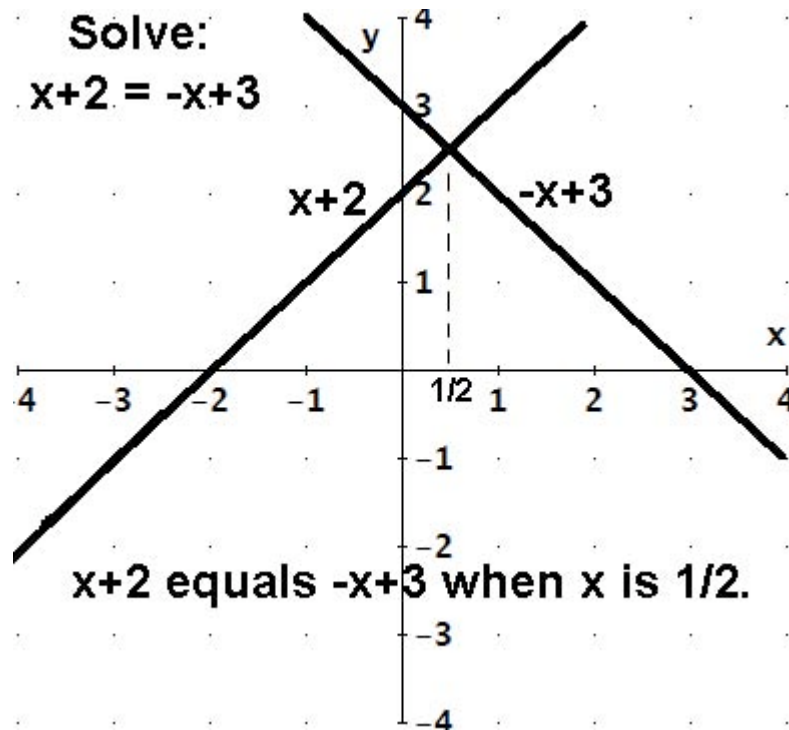
For example, using the “*alternative teaching strategy*” for graphical method of solving a linear equation system, the teacher may display a set of co – ordinate axes on the over head. A point $(1/2, 5/2)$ may be plotted on the graph. The students are asked to write the equations of two lines that pass through the point. The students participating in the discussion will come out with different equations of lines.

For Example,

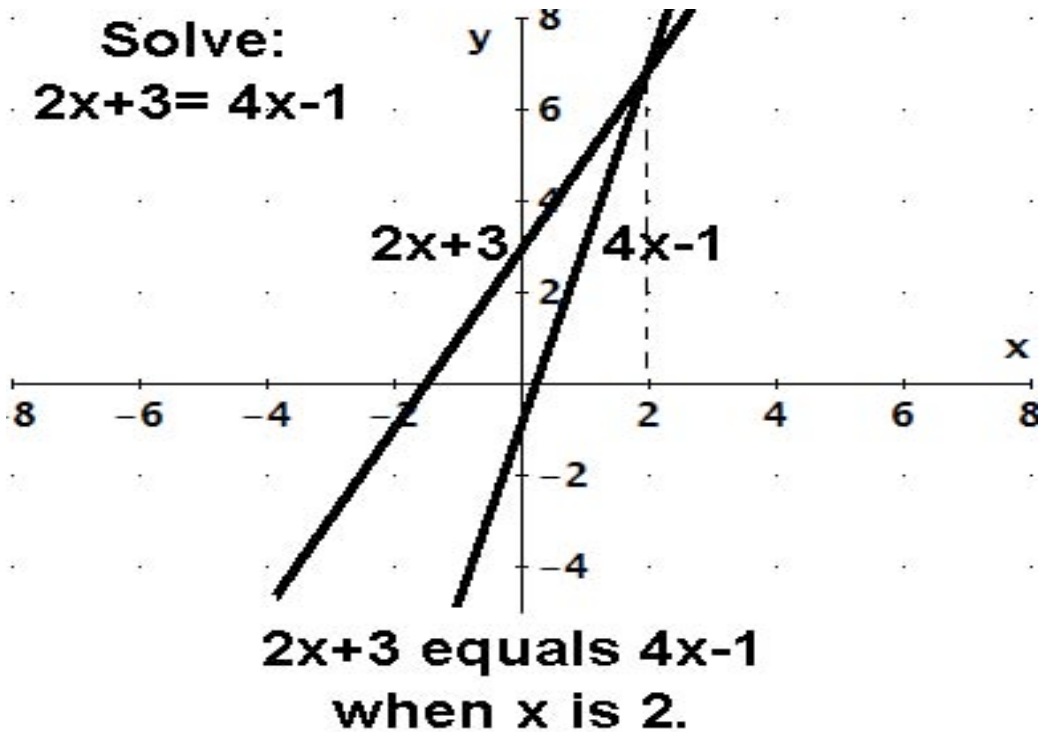
$$y = x + 2, y = -x + 3$$

x	2	1	-2
y	4	3	0

x	1	3	5
y	2	0	-2



Example # 2:



$$2x + 3 = 4x - 1$$

The two expressions " $2x + 3$ " and " $4x - 1$ " don't mean the same thing but there may be a time when they are equal. They are equal, the lines cross, when x is 2, and only when x is 2.

The solution is 2.

An effective "***Interdisciplinary general education program***" regularly reviews its goals, its curriculum, and the courses which it offers. In addition, the program has a process for regularly monitoring progress toward achieving integrative goals and outcomes, curriculum, and courses.

The students are asked to plot the graph of these equations and to their surprise both lines pass through the point $(1/2, 5/2)$. Questions for students: Is there any other point where these two lines intersect or pass through? The prompt answer is: No. thus the response is elaborated to define the system of equations. The system of equations with a unique solution is called "***Consistent system.***"

Like wise for the "***Inclusion strategies***", have students write each equation in a system with a pencil of different colors. Then tell them to graph each

equation in the corresponding color. Suggest that they use a third color to draw the point of intersection and to write the co – ordinates of that point.

“Enrichment” is the most effective strategy for students to research on a problem. For Example, An increasing number of American households are choosing to obtain their television reception through a satellite dish rather than a cable. A question then arises, is it always beneficial for the household to get a satellite connection over cable? The answer to this depends on certain time factors. For instance, in a locality one company markets satellite service for an initial fee of \$300; this included the purchase of dish and installation, and then charges \$20 per month for programming. A cable television company in the same area requires an initial fee of \$14 for installation and \$31 per month thereafter. Ask students to solve system of equations

$$C = 300 + 20 t \text{ (For Dish Network company)}$$

$$C = 14 + 31 t \text{ (For Cable - network company)}$$

When the students solve the system, they get $t = 26$ and $C = 820$.it means the cost of two networks for 26 months is the same i.e. \$820.Now if a person wants to reside in that area for only 20 months (less than 26 months) or forever, which type of service is better?

Research showed that cable company will charge the person $\$ (14 + 31 \times 20) = \634 while dish company will charge the person $\$ (300 + 20 \times 20) = \700 for 20 months. So the conclusion of research work is cable – network is better for the time period of less than 26 months while dish network is better for time period of more than 26 months.

CLASS ACTIVITIES

(LESSON PLANS)

OBJECTIVES

- a) Solving equations by adding and subtracting
- b) Solving equations by multiplying and dividing
- c) Solving two-step equations
- d) Solving multisteps equations
- e) Using distributive property
- f) Using formulas and literal equations

CONTENT /PERFORMANCE

- a) Simplify multi-step problems involving positive exponents
- b) Perform calculations requiring multiple procedures and operations involving whole numbers, decimals, fractions and percents.
- c) Represent given information as expressions, equations, inequalities, systems of equations, systems of inequalities, or matrices.
- d) Determine the number of solutions that exist for a given system of equations.
- e) Classify and describe system of equations.
- f) Demonstrate recognition of the graphic representations of system of linear equations and determine their solutions.

PA Standards (refer to standards alignment):

2.1 A Page 4; 2.2 A Page 6; 2.6 C Page 34; 2.8 D Page 36; 2.8 F, 2.8 G, 2.8 H Page 38.

Textual References and Structural Lessons: Holt Algebra 1: Chapter1: Page 4 -49;
Chapter 2: Page 54- 108; Chapter 3: pp 135 – 153; Chapter 7; Pp 318-337
Technology: Page: 316-D.

Calculator keystroke guide, P – 789

Portfolio activities: pp – 325, 337

Curriculum Resources:

Chapter – pp 116-120

Other Instructional Resources/Technology:

Lesson Activities, pp 14-16

Cooperative – Learning Activities, pp.14-16

Student Technology Guide, pp. 14 – 16

Mathepedia CD- ROM

Writing Activities for your Portfolio, pp.7-9

Activities Online: Egyptian Multiplication

www.go.hrw.com(Keyword: MA1 Egyptian)

Activities Online: Oscar at Bat, www.go.hrw.com (Keyword: MA 1 Oscar at Bat).

Sample Assessment Item:

Simplify:

1. $(-3x + 9y) - (2x + 14y)$

- $-5x - 5y$
- $-5x + 23y$
- $-x - 5y$
- $-x + 23y$

2. $(12x - 4y + 10) - 2(5x - 2y)$

- $22x + 10$
- $2x + 10$
- $2x - 8y + 10$
- $22x - 8y + 10$

3. Solve for x: $-6x + 7 = 5x + 62$

- -5
- 5
- 55
- 71

4. Romeo is constructing a border around his vegetable patch. He wants the length to be four times the width. He has only 100 feet of border material. Can he build a border that is the length and the width he wants and that uses all of his border material? Show your work and explain your solution.

Equations are the building blocks of algebra and help the students in modeling arithmetical problems to algebraic problems by involving equations.

For Example;

A boy has 28 coins in all, in dimes and nickels and when he counted the total value of coins, it came out to be \$2.00. How many dimes and nickels the boy has?

If this problem is solved arithmetically using hit and trail method, it is time consuming and boring because he will have to start at some dimes and nickel, say 5 dimes and 23 nickels and check the value, 6 dimes and 22 nickels and so on. Here algebra helps you. Let us assume that the boy has x number of dimes and y number of nickels. Then according to the data in question

$$x + y = 28 \dots\dots\dots (1)$$

And

$$10x + 5y = 200 \dots\dots\dots (2)$$

Equation (2) can be written as an equivalent equation by dividing throughout by 5. So system of equations is reduced to:

$$x + y = 28 \dots\dots\dots (1)$$

And

$$2x + y = 40 \dots\dots\dots (2)$$

Solving the system by any convenient method: $x = 12$, $y = 16$ on check, we find that the boy has 12 dimes and 16 nickels. So there are applications of equations to connect them to real world situations.

The same problem could have been solved by using one variable too. Let us assume the number of dimes with the boy be X . So number of nickels will be $(28 - x)$. For the value equation $10x + 5(28 - x) = 200$. Solving for X , we get the solution $X = 12$. Therefore, the number of dimes is equal to 12 and the number of nickels is $(28 - 12)$ i.e. 16.

References:

1. Holt Algebra
2. www.aaas.org
3. NCTM Standards – www.enc.org/online/nctm/280tocl.html
4. [http:// home.ptd.net/pctm](http://home.ptd.net/pctm)
5. www.hrw.com
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